Electrodynamics & Relativity

Week 1: Mathematical Language of Electrodynamics

Scalar & vector fields, Differential operators, line/surface/volume Integrals, Gauss's & Stokes' theorem, Coordinate systems, Dirac delta function

Abbas Ali Saberi

School of Science, Constructor University, Bremen, Germany

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Main idea. Electrodynamics is a theory of fields. A field has a value at every point in space and, in general, at every time. This week develops the geometric language used to read such fields: how a field changes locally, where it has sources, how it circulates, how it flows through a boundary, and how local information is converted into measurable integral statements.
The central pattern is

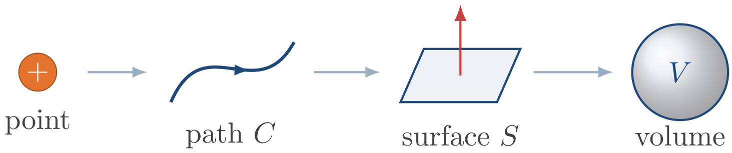
$$\boxed{\text{local field structure} \quad \Longleftrightarrow \quad \text{flux and circulation on boundaries}}$$

Gradient, divergence, curl, the Laplacian, line and surface integrals, Gauss's theorem, Stokes' theorem, and the Dirac delta will later appear directly inside Maxwell's theory.

1. Position, vectors, and fields

We begin with the geometry of three-dimensional space.

Throughout electrodynamics, different operations naturally involve points, paths, surfaces, and volumes. The following picture gives the geometric hierarchy that will repeatedly reappear in the course.



Point sources, line integrals, surface fluxes, and volume integrals are different geometric parts of the same field theory.

A point is specified by the position vector

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}.$$

Here:

- x, y, and z are Cartesian coordinates,
- $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors along the Cartesian axes,
- $\mathbf{r}$ is the position vector.

The magnitude of $\mathbf{r}$ is

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

The radial unit vector is

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}.$$

A unit vector has magnitude one:

$$|\hat{\mathbf{r}}| = 1.$$

1.1 Dot product

For two vectors $\mathbf{A}$ and $\mathbf{B}$, the dot product is

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta,$$

where:

- $A = |\mathbf{A}|$,
- $B = |\mathbf{B}|$,
- θ is the angle between the vectors.

The dot product measures how much one vector points along another.
If the vectors are perpendicular,

$$\mathbf{A} \cdot \mathbf{B} = 0.$$

This operation will later appear in work and flux integrals.

1.2 Cross product

The cross product is

$$\mathbf{A} \times \mathbf{B} = AB\sin\theta\,\hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$.
Its direction follows the right-hand rule.
The cross product will later appear in the magnetic Lorentz force,

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B},$$

and, in vacuum, in the Poynting vector,

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}.$$

2. Scalar and vector fields

A **field** assigns a physical quantity to every point in space.
In general, a field may also depend on time.

2.1 Scalar fields

A scalar field assigns one number to every position:

$$f = f(\mathbf{r}).$$

A central example is the electric potential,

$$\Phi = \Phi(\mathbf{r}),$$

where Φ is measured in volts.

For a point charge q at the origin,

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}.$$

Here q is the electric charge and ε_0 is the vacuum permittivity.

2.2 Vector fields

A vector field assigns a vector to every point:

$$\mathbf{A} = \mathbf{A}(\mathbf{r}).$$

In Cartesian coordinates,

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}.$$

The quantities A_x, A_y, and A_z are the Cartesian components of the vector field.

The electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are vector fields.

For a point charge,

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

2.3 Space- and time-dependent fields

In electrostatics, the fields do not change with time, so we often write only their spatial dependence:

$$\Phi = \Phi(\mathbf{r}), \qquad \mathbf{E} = \mathbf{E}(\mathbf{r}).$$

In full electrodynamics, fields may change both from point to point and with time:

$$\Phi = \Phi(\mathbf{r}, t), \qquad \mathbf{E} = \mathbf{E}(\mathbf{r}, t), \qquad \mathbf{B} = \mathbf{B}(\mathbf{r}, t).$$

Here t denotes time.

A spatial partial derivative such as

$$\frac{\partial E_x}{\partial x}$$

asks how the x component of the electric field changes when we move in the x direction while keeping y, z, and t fixed.

A time derivative such as

$$\frac{\partial \mathbf{B}}{\partial t}$$

asks how the magnetic field changes with time at one fixed position $\mathbf{r}$.

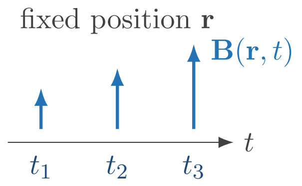
Physical meaning

The distinction between spatial and temporal change is important.
At a fixed point,

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \text{local change of } \mathbf{B} \text{ with time}}$$

Later, Faraday's law will connect this local time variation to the curl of the electric field:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$



At the same fixed position $\mathbf{r}$, the field magnitude shown here increases with time.

If the observation point itself moves with velocity $\mathbf{v}$, the measured field value can change for **two distinct reasons**.

First, the field may have an **explicit time dependence** even at a fixed position. Second, the observer may move through a **spatially nonuniform field**, so that the value changes simply because different points in space carry different values of the field.

For a scalar field $f(\mathbf{r}, t)$, the total rate of change seen by the moving observer is

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the velocity of the observation point.

This decomposition is often called the **total derivative along the trajectory**. It will reappear later when we discuss moving charges, induced fields, and conductors in motion.

3. The differential operator ∇

The symbol ∇ is called **nabla** or **del**.

In Cartesian coordinates,

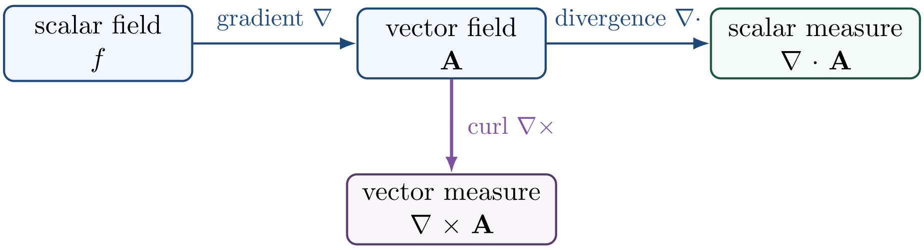
$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.$$

The symbol $\partial/\partial x$ means a partial derivative with respect to x, while the other independent variables are kept fixed.

Depending on how ∇ acts, it produces four important operations:

gradient, divergence, curl, Laplacian.

These four objects describe different aspects of how a field changes in space.



∇ is a differential operator. Its meaning depends on how it acts: gradient maps a scalar to a vector, divergence maps a vector to a scalar, and curl maps a vector to a vector. The Laplacian ∇^2 involves a second spatial derivative.

4. Gradient

The gradient acts on a scalar field and produces a vector field.

For $f = f(x, y, z)$,

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}.$$

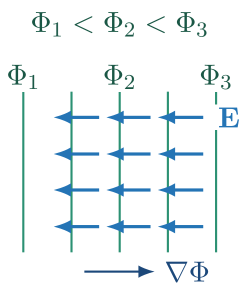
The gradient points in the direction in which f increases most rapidly.

Its magnitude gives the rate of this increase per unit distance.

For electrostatics,

$$\boxed{\mathbf{E} = -\nabla\Phi}$$

where Φ is the electric potential and $\mathbf{E}$ is the electric field.
The electric field therefore points toward decreasing potential.



Equipotential lines are perpendicular to $\mathbf{E}$. The gradient $\nabla\Phi$ points toward increasing potential, while $\mathbf{E} = -\nabla\Phi$ points toward decreasing potential.

The figure gives a useful geometric picture. The green lines represent equipotential surfaces in this two-dimensional cross-section. The gradient is normal to them and points toward increasing potential. The electric field points in the opposite direction.

4.1 Example: a uniform electric field

Example

Consider

$$\Phi(x, y, z) = -E_0 x,$$

where E_0 is a positive constant with units of electric field.
Then

$$\nabla\Phi = -E_0\hat{\mathbf{x}}.$$

Therefore,

$$\mathbf{E} = -\nabla\Phi = E_0\hat{\mathbf{x}}.$$

The potential decreases as x increases, while the electric field points in the positive x direction.

4.2 Gradient of $1/r$

An important relation is

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{\mathbf{r}}}{r^2}, \qquad r \neq 0.$$

Derivation

Since

$$r = \sqrt{x^2 + y^2 + z^2},$$

we have

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

Using the chain rule,

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{x}{r^3}.$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3},$$

and

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{r^3}.$$

Therefore,

$$\nabla\left(\frac{1}{r}\right) = -\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r^3} = -\frac{\hat{\mathbf{r}}}{r^2}.$$

For the point-charge potential,

$$\Phi = \frac{q}{4\pi\varepsilon_0 r},$$

this gives

$$\mathbf{E} = -\nabla\Phi = \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Thus the familiar Coulomb field follows directly from the spatial variation of the potential.

5. Divergence

The divergence acts on a vector field and produces a scalar field.

For

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}},$$

the divergence is

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

The divergence measures the net outward flux generated locally per unit volume. Positive divergence corresponds to local source behavior; negative divergence corresponds to local sink behavior.

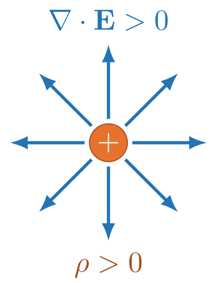
$$\boxed{\nabla \cdot \mathbf{A} > 0 \quad \Longrightarrow \quad \text{local source}}$$

$$\boxed{\nabla \cdot \mathbf{A} < 0 \quad \Longrightarrow \quad \text{local sink}}$$

In electrostatics,

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}}$$

so electric charge density ρ acts as a source of the electric field.



A positive charge is a source of $\mathbf{E}$.

This picture should not be interpreted as saying that field lines are physical strings. Field lines visualize the direction of a vector field and, through their drawn density, can also suggest relative field strength.

5.1 Example

Consider

$$\mathbf{A} = \alpha\left(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\right),$$

where α is a constant.

Then

$$\nabla \cdot \mathbf{A} = \alpha + \alpha + \alpha = 3\alpha.$$

For $\alpha > 0$, the field points outward and has positive divergence.

5.2 Local meaning

The divergence can be defined as outward flux per unit volume:

$$\nabla \cdot \mathbf{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{\partial(\Delta V)} \mathbf{A} \cdot d\mathbf{S}.$$

Here:

- ΔV is a small volume,

- $\partial(\Delta V)$ is its closed boundary,
- $d\mathbf{S}$ is an outward-directed surface element.

The divergence therefore compares the amount of field leaving a small volume with the size of that volume.

6. Curl

The curl acts on a vector field and produces another vector field.

For

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}},$$

the curl in Cartesian coordinates is

$$\begin{aligned}\nabla \times \mathbf{A} = &\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} \\ &+ \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} \\ &+ \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}.\end{aligned}$$

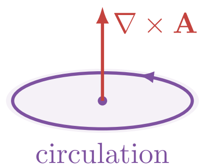
The curl measures the local circulation of a vector field. Its direction is normal to the plane of positive circulation, fixed by the right-hand rule, while its magnitude gives the circulation per unit area in the infinitesimal limit.

$\boxed{\nabla \times \mathbf{A}} \iff$ local circulation of $\mathbf{A}$

More precisely, the component of the curl normal to a small surface is the circulation per unit area:

$$\hat{\mathbf{n}} \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_{\partial(\Delta S)} \mathbf{A} \cdot d\mathbf{l}.$$

Here $\hat{\mathbf{n}}$ is the unit vector normal to the surface, ΔS is its area, and $d\mathbf{l}$ is the directed line element along its boundary.
The direction follows the right-hand rule.



The circulation lies in the local plane. The curl points perpendicular to that plane.

6.1 Example: vector potential of a uniform magnetic field

Consider

$$\mathbf{A} = \frac{B_0}{2}\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right),$$

where B_0 is a constant.

The Cartesian components are

$$A_x = -\frac{B_0 y}{2}, \qquad A_y = \frac{B_0 x}{2}, \qquad A_z = 0.$$

The only nonzero component of the curl is the z component:

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\mathbf{z}}.$$

Since

$$\frac{\partial A_y}{\partial x} = \frac{B_0}{2}, \qquad \frac{\partial A_x}{\partial y} = -\frac{B_0}{2},$$

we obtain

$$\nabla \times \mathbf{A} = \left(\frac{B_0}{2} + \frac{B_0}{2} \right) \hat{\mathbf{z}} = B_0 \hat{\mathbf{z}}.$$

Therefore,

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} = B_0 \hat{\mathbf{z}}}.$$

This $\mathbf{A}$ is a magnetic vector potential for the uniform field $\mathbf{B} = B_0 \hat{\mathbf{z}}$. The relation $\mathbf{B} = \nabla \times \mathbf{A}$ will become central in magnetostatics.

6.2 Curl in electrodynamics: Faraday induction

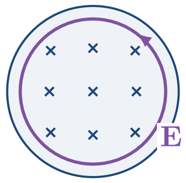
One of the most important appearances of curl is Faraday's law.

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Here:

- $\mathbf{E}$ is the electric field,
- $\mathbf{B}$ is the magnetic field,
- t is time,
- $\partial \mathbf{B}/\partial t$ is the local rate of change of the magnetic field.

A changing magnetic field therefore produces a circulating electric field.
The minus sign gives the direction required by Lenz's law.



An increasing magnetic field into the page induces a counterclockwise electric field.

The figure shows the physical meaning of the minus sign.

If $\mathbf{B}$ points into the page and increases in magnitude, the induced $\mathbf{E}$ circulates counterclockwise. The induced circulation acts in the direction required to oppose the increase of the magnetic flux.

7. Laplacian

The Laplacian of a scalar field is

$$\nabla^2 f = \nabla \cdot (\nabla f).$$

In Cartesian coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The Laplacian is the **divergence of the gradient**. It measures the net local curvature of a scalar field: it compares the value at one point with the values in the nearby region.
A useful intuition is:

- if the point sits in a **local valley**, then nearby gradient vectors tend to point outward, so

$$\nabla^2 f > 0;$$

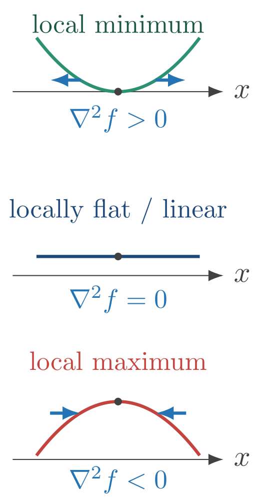
- if the point sits on a **local hill**, then nearby gradient vectors tend to point inward, so

$$\nabla^2 f < 0;$$

- if the field is locally **flat** or changes only linearly, then

$$\nabla^2 f = 0.$$

So the Laplacian measures how the slope changes from place to place. In electrostatics, this local curvature is tied directly to charge density through Poisson's equation.



valley → positive Laplacian,
flat region → zero Laplacian,
hill → negative Laplacian.

In electrostatics,

$$\boxed{\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}}$$

This is Poisson's equation.
In a region where $\rho = 0$,

$$\boxed{\nabla^2 \Phi = 0.}$$

This is Laplace's equation.

7.1 Example

Let

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Then

$$\nabla^2 f = 2 + 2 + 2 = 6.$$

8. Differential operators at a glance

Operation	Input → output	Geometric meaning	Electrodynamics
∇f	scalar → vector	fastest spatial increase	$\mathbf{E} = -\nabla\Phi$
$\nabla \cdot \mathbf{A}$	vector → scalar	local source or sink	$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$
$\nabla \times \mathbf{A}$	vector → vector	local circulation	$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$
$\nabla^2 f$	scalar → scalar	local spatial curvature	$\nabla^2\Phi = -\rho/\varepsilon_0$

9. Important vector identities

For a scalar field f and vector fields $\mathbf{A}$ and $\mathbf{B}$,

$$\nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A}),$$

$$\nabla \times (f\mathbf{A}) = (\nabla f) \times \mathbf{A} + f(\nabla \times \mathbf{A}),$$

and

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

Three identities will be especially important. In the third identity, $\nabla^2\mathbf{A}$ denotes the vector Laplacian; in Cartesian coordinates it acts on each component of $\mathbf{A}$.

$$\boxed{\nabla \times (\nabla f) = 0}$$

$$\boxed{\nabla \cdot (\nabla \times \mathbf{A}) = 0}$$

$$\boxed{\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}}$$

These identities have immediate physical consequences.

If

$$\mathbf{E} = -\nabla\Phi,$$

then

$$\nabla \times \mathbf{E} = 0.$$

If

$$\mathbf{B} = \nabla \times \mathbf{A},$$

then

$$\nabla \cdot \mathbf{B} = 0.$$

These relations will later explain why electrostatics admits a scalar potential and why magnetic fields can be described by a vector potential.

9.1 Derivation of $\nabla \times (\nabla f) = 0$

Derivation

Consider the z component:

$$[\nabla \times (\nabla f)]_z = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

For a sufficiently smooth function,

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}.$$

The two terms therefore cancel.
The same argument applies to the other components, giving

$$\nabla \times (\nabla f) = 0.$$

9.2 Derivation of $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

Derivation

Expanding the divergence gives

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{A}) &= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\\ &+ \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\\ &+ \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).\end{aligned}$$

Every mixed derivative appears twice with opposite sign. The terms cancel pairwise.
Therefore,

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

10. Line integrals

A line integral adds the component of a vector field along a path.

Let the path C be described by

$$\mathbf{r} = \mathbf{r}(\lambda),$$

where λ is a parameter that labels points along the path.

The differential displacement is

$$d\mathbf{l} = d\mathbf{r} = \frac{d\mathbf{r}}{d\lambda}\, d\lambda.$$

For a vector field $\mathbf{A}$,

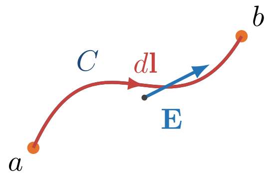
$$\int_C \mathbf{A} \cdot d\mathbf{l}$$

is the line integral of $\mathbf{A}$ along C.

The dot product selects the component of $\mathbf{A}$ tangent to the path.
In electrostatics,

$$\Phi(\mathbf{r}_b) - \Phi(\mathbf{r}_a) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}.$$

Thus the potential difference is obtained by accumulating the tangential electric field along the path.



Only the component of $\mathbf{E}$ parallel to $d\mathbf{l}$ contributes to $\mathbf{E} \cdot d\mathbf{l}$.

Derivation

For a small displacement,

$$d\Phi = \nabla\Phi \cdot d\mathbf{r}.$$

Since

$$\mathbf{E} = -\nabla\Phi,$$

we obtain

$$d\Phi = -\mathbf{E} \cdot d\mathbf{r}.$$

Integrating between two points gives

$$\Phi(\mathbf{r}_b) - \Phi(\mathbf{r}_a) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}.$$

10.1 Fundamental theorem for gradients

The relation between a gradient and a line integral is general.

For any sufficiently smooth scalar field $f(\mathbf{r})$,

$$\boxed{\int_a^b \nabla f \cdot d\mathbf{l} = f(\mathbf{r}_b) - f(\mathbf{r}_a)}$$

where $\mathbf{r}_a$ and $\mathbf{r}_b$ are the initial and final positions.

The result depends only on the endpoints, not on the path taken between them.

Derivation

Let the path be described by $\mathbf{r} = \mathbf{r}(\lambda)$, where λ is a parameter.
Along the path,

$$df = \nabla f \cdot d\mathbf{r}.$$

With the directed line element $d\mathbf{l} = d\mathbf{r}$ along the path,

$$df = \nabla f \cdot d\mathbf{l}.$$

Integrating from the starting point a to the final point b gives

$$\int_a^b df = \int_a^b \nabla f \cdot d\mathbf{l}.$$

The left-hand side is simply

$$f(\mathbf{r}_b) - f(\mathbf{r}_a).$$

Therefore,

$$\int_a^b \nabla f \cdot d\mathbf{l} = f(\mathbf{r}_b) - f(\mathbf{r}_a).$$

10.2 Conservative fields and path independence

A vector field is called **conservative** if it can be written as the gradient of a scalar potential, up to a conventional sign:

$$\mathbf{A} = -\nabla\Phi.$$

For such a field,

$$\int_a^b \mathbf{A}\cdot d\mathbf{l} = -\left[\Phi(\mathbf{r}_b) - \Phi(\mathbf{r}_a)\right].$$

Therefore, the line integral is independent of the path.

For a conservative field,

$$\boxed{\oint_C \mathbf{A}\cdot d\mathbf{l} = 0}$$

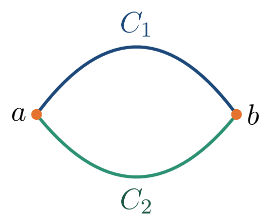
for every closed path C.
Also,

$$\boxed{\mathbf{A} = -\nabla\Phi \quad\Longrightarrow\quad \nabla\times\mathbf{A} = 0.}$$

Conversely, in a simply connected region,

$$\boxed{\nabla\times\mathbf{A} = 0 \quad\Longrightarrow\quad \mathbf{A}\text{ is conservative.}}$$

A **simply connected region** has no holes that obstruct continuous shrinking of a closed loop to a point.



same endpoints
same potential difference

For a conservative field, different paths give the same line integral.

In electrostatics,

$$\mathbf{E} = -\nabla\Phi$$

and therefore

$$\nabla\times\mathbf{E} = 0, \qquad \oint_C \mathbf{E}\cdot d\mathbf{l} = 0.$$

These relations will be central when we introduce the electrostatic potential in detail.

11. Surface integrals and flux

A surface integral measures how much of a vector field passes through a surface.

For a small surface element of area dS with unit normal $\hat{\mathbf{n}}$, we define the directed surface element

$$d\mathbf{S} = \hat{\mathbf{n}}\,dS.$$

Here dS is the scalar area of the surface element, while $d\mathbf{S}$ is a vector perpendicular to the surface.

The flux of a vector field $\mathbf{A}$ through a surface S is

$$\Phi_A = \int_S \mathbf{A}\cdot d\mathbf{S},$$

where Φ_A denotes the flux of $\mathbf{A}$ through S. The subscript distinguishes this flux symbol from the electrostatic potential Φ.

For a closed surface,

$$\Phi_A = \oint_S \mathbf{A} \cdot d\mathbf{S}$$

is the total outward flux.

Physical meaning

For one small surface element,

$$\boxed{\mathbf{A} \cdot d\mathbf{S} = A\, dS \cos\theta}$$

where:

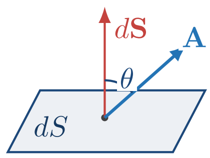
- $A = |\mathbf{A}|$ is the magnitude of the vector field,
- dS is the area of the small surface element,
- θ is the angle between $\mathbf{A}$ and the outward normal $\hat{\mathbf{n}}$.

Only the component of $\mathbf{A}$ perpendicular to the surface contributes:

$$A_\perp = A \cos\theta.$$

Therefore,

$$\mathbf{A} \cdot d\mathbf{S} = A_\perp\, dS.$$



Only the component of $\mathbf{A}$ along the surface normal contributes to the flux.

11.1 Electric flux from a point charge

Consider a sphere of radius R centered on a point charge q.

The electric field is

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 R^2}\hat{\mathbf{r}}.$$

The surface element is radial:

$$d\mathbf{S} = \hat{\mathbf{r}}\, dS.$$

Thus

$$\mathbf{E} \cdot d\mathbf{S} = \frac{q}{4\pi\varepsilon_0 R^2}\, dS.$$

Since the area of the sphere is $4\pi R^2$,

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\varepsilon_0}.$$

The total flux is independent of R.

This fact will become Gauss's law.

12. Volume integrals

A volume integral adds a quantity throughout a three-dimensional region.

For a scalar field f,

$$\int_V f\,dV$$

means integration over the volume V.

In electrodynamics, an important scalar field is the charge density

$$\rho = \rho(\mathbf{r}),$$

where ρ is charge per unit volume.

Its SI unit is $\mathrm{C/m^3}$.

The total charge inside a volume V is

$$Q = \int_V \rho(\mathbf{r})\,dV.$$

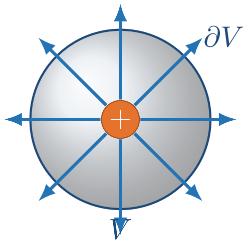
13. Gauss's divergence theorem

Gauss's theorem connects local divergence inside a volume with total flux through its boundary.

For a volume V with closed boundary ∂V,

$$\boxed{\int_V \nabla \cdot \mathbf{A}\,dV = \oint_{\partial V} \mathbf{A} \cdot d\mathbf{S}}$$

The left-hand side adds all local sources inside the volume.
The right-hand side measures the total field flowing out through the boundary.



Sources inside V determine the outward flux through ∂V.

13.1 Why internal surfaces cancel

Derivation

Imagine dividing V into many very small cells.
For two neighbouring cells, their common surface has opposite outward normals.
The flux leaving one cell through that internal surface is therefore equal and opposite to the flux entering the neighbouring cell.
All internal surface contributions cancel.
Only the flux through the external boundary ∂V remains.
In the continuum limit, the sum of all local divergences becomes

$$\int_V \nabla \cdot \mathbf{A}\,dV,$$

and the remaining boundary flux becomes

$$\oint_{\partial V} \mathbf{A} \cdot d\mathbf{S}.$$

13.2 From integral Gauss law to differential Gauss law

Later, Gauss's law will be

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dV.$$

Gauss's theorem converts the left-hand side:

$$\int_V \nabla \cdot \mathbf{E} \, dV = \frac{1}{\varepsilon_0} \int_V \rho \, dV.$$

Therefore,

$$\int_V \left(\nabla \cdot \mathbf{E} - \frac{\rho}{\varepsilon_0} \right) dV = 0.$$

Since this must hold for every volume,

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}}$$

at every regular point.

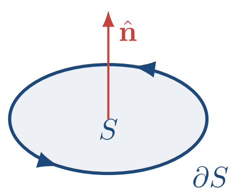
14. Stokes' theorem

Stokes' theorem connects local curl over a surface with total circulation around its boundary.

For a surface S with boundary ∂S,

$$\boxed{\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{l}}$$

The surface normal and the positive direction around the boundary are connected by the right-hand rule.



Curl through S equals circulation around ∂S.

14.1 Why the theorem works

Derivation

Divide S into many small surface elements.
Each element has a small circulation around its boundary.
Every internal edge is traversed twice in opposite directions by neighbouring elements.
The internal contributions cancel.
Only the circulation around the external boundary remains.
In the continuum limit,

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{l}.$$

14.2 Connection to Faraday's law

For a fixed surface S, Faraday's law will later be written as

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}.$$

Using Stokes' theorem,

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}.$$

Therefore,

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

which is the differential form of Faraday's law.

Physical meaning

The three fundamental calculus relations now form one geometric hierarchy:

$$\boxed{\begin{array}{rcll} \displaystyle\int_a^b \nabla f \cdot d\mathbf{l} & = & f(b) - f(a) & \text{endpoints,} \\ \displaystyle\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} & = & \displaystyle\oint_{\partial S} \mathbf{A} \cdot d\mathbf{l} & \text{boundary of a surface,} \\ \displaystyle\int_V (\nabla \cdot \mathbf{A})\, dV & = & \displaystyle\oint_{\partial V} \mathbf{A} \cdot d\mathbf{S} & \text{boundary of a volume.} \end{array}}$$

A derivative inside a region is converted into information on its boundary. This local–global relation is one of the organizing ideas of Maxwell theory.

15. Coordinate systems and symmetry

The laws of physics do not depend on the coordinate system.

The difficulty of a calculation often does.

A good coordinate system follows the symmetry of the physical problem. When the coordinates follow the geometry, many derivatives vanish and the field equations become much simpler before any detailed calculation begins.

Coordinates	**Natural symmetry**	**Electrodynamics example**
Cartesian	planar	parallel plates, rectangular boundaries
Cylindrical	axial	long wire, cylinder, coaxial cable
Spherical	radial	point charge, charged sphere, multipoles

15.1 Orthogonal coordinates and scale factors

In a general orthogonal coordinate system (q_1, q_2, q_3),

$$d\mathbf{l} = h_1 dq_1\,\hat{\mathbf{e}}_1 + h_2 dq_2\,\hat{\mathbf{e}}_2 + h_3 dq_3\,\hat{\mathbf{e}}_3.$$

Here:

- q_i are coordinates,
- $\hat{\mathbf{e}}_i$ are local unit vectors,
- h_i are scale factors.

The volume element is

$$dV = h_1 h_2 h_3\, dq_1 dq_2 dq_3.$$

16. Cartesian coordinates

Cartesian coordinates are (x, y, z).
The line element is

$$d\mathbf{l} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz,$$

and the volume element is

$$dV = dx\,dy\,dz.$$

For a scalar field f,

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}.$$

For

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}},$$

the divergence is

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

The scalar Laplacian is

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

17. Cylindrical coordinates

Cylindrical coordinates are

$$(\varrho, \phi, z).$$

We use ϱ for cylindrical radius so that ρ remains available for electric charge density.
Here:

- ϱ is the distance from the z axis,
- ϕ is the azimuthal angle,
- z is the height.

The relation to Cartesian coordinates is

$$x = \varrho\cos\phi, \qquad y = \varrho\sin\phi.$$

The line element is

$$d\mathbf{l} = \hat{\boldsymbol{\varrho}}\, d\varrho + \hat{\boldsymbol{\phi}}\, \varrho\, d\phi + \hat{\mathbf{z}}\, dz.$$

The volume element is

$$dV = \varrho\, d\varrho\, d\phi\, dz.$$

The gradient is

$$\nabla f = \hat{\boldsymbol{\varrho}}\frac{\partial f}{\partial \varrho} + \hat{\boldsymbol{\phi}}\frac{1}{\varrho}\frac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}.$$

For

$$\mathbf{A} = A_\varrho\hat{\boldsymbol{\varrho}} + A_\phi\hat{\boldsymbol{\phi}} + A_z\hat{\mathbf{z}},$$

the divergence is

$$\nabla\cdot\mathbf{A} = \frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho A_\varrho\right) + \frac{1}{\varrho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}.$$

The scalar Laplacian is

$$\nabla^2 f = \frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\frac{\partial f}{\partial\varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

The curl is

$$\begin{aligned}\nabla\times\mathbf{A} = \hat{\boldsymbol{\varrho}}&\left[\frac{1}{\varrho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\\ &+ \hat{\boldsymbol{\phi}}\left[\frac{\partial A_\varrho}{\partial z} - \frac{\partial A_z}{\partial\varrho}\right]\\ &+ \hat{\mathbf{z}}\frac{1}{\varrho}\left[\frac{\partial}{\partial\varrho}\left(\varrho A_\phi\right) - \frac{\partial A_\varrho}{\partial\phi}\right].\end{aligned}$$

In physical problems, symmetry usually removes many of these terms before the formula is evaluated.

Surface elements. The directed surface element depends on which coordinate is held constant. For a cylindrical surface $\varrho = \text{const.}$,

$$\boxed{d\mathbf{S} = \hat{\boldsymbol{\varrho}}\,\varrho\, d\phi\, dz}$$

For a surface $\phi = \text{const.}$,

$$d\mathbf{S} = \hat{\boldsymbol{\phi}}\, d\varrho\, dz.$$

For a horizontal surface $z = \text{const.}$,

$$\boxed{d\mathbf{S} = \hat{\mathbf{z}}\, \varrho\, d\varrho\, d\phi}$$

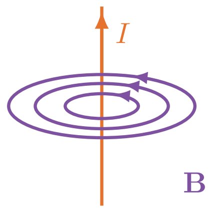
The direction shown above is the direction of increasing coordinate. For a closed surface, the sign must always be chosen so that $d\mathbf{S}$ points outward.

17.1 Why cylindrical coordinates are useful

For an infinitely long straight wire, symmetry tells us that the magnetic field cannot depend on z or ϕ.
It can only depend on the distance ϱ from the wire.
Its direction must be azimuthal:

$$\boxed{\mathbf{B} = B(\varrho)\hat{\boldsymbol{\phi}}}$$

The coordinate system already captures most of the physics before any calculation is done.



The field of a long straight wire has cylindrical symmetry.

18. Spherical coordinates

Spherical coordinates are

$$(r, \theta, \phi).$$

Here:

- r is the distance from the origin,
- θ is the polar angle measured from the positive z axis,
- ϕ is the azimuthal angle.

The Cartesian coordinates are

$$x = r \sin\theta \cos\phi,$$

$$y = r \sin\theta \sin\phi,$$

$$z = r \cos\theta.$$

The line element is

$$d\mathbf{l} = \hat{\mathbf{r}}\, dr + \hat{\boldsymbol{\theta}}\, r\, d\theta + \hat{\boldsymbol{\phi}}\, r \sin\theta\, d\phi.$$

The volume element is

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

The gradient is

$$\nabla f = \hat{\mathbf{r}} \frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi}.$$

For

$$\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}},$$

the divergence is

$$\begin{aligned}\nabla \cdot \mathbf{A} = &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right)\\ &+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right)\\ &+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}.\end{aligned}$$

The scalar Laplacian is

$$\begin{aligned}\nabla^2 f = &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\\ &+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)\\ &+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.\end{aligned}$$

The curl is

$$\begin{aligned}\nabla \times \mathbf{A} = &\,\hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right)-\frac{\partial A_\theta}{\partial\phi}\right]\\ &+\hat{\boldsymbol{\theta}}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}\left(rA_\phi\right)\right]\\ &+\hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right)-\frac{\partial A_r}{\partial\theta}\right].\end{aligned}$$

In physical problems, spherical symmetry or axial symmetry usually removes many of these terms.

Surface elements. For a spherical surface $r = \text{const.}$,

$$\boxed{d\mathbf{S} = \hat{\mathbf{r}}\, r^2 \sin\theta\, d\theta\, d\phi}$$

For a surface $\theta = \text{const.}$,

$$d\mathbf{S} = \hat{\boldsymbol{\theta}}\, r\sin\theta\, dr\, d\phi.$$

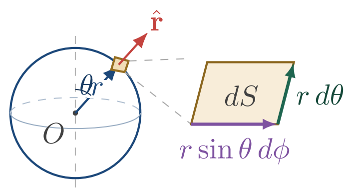
For a surface $\phi = \text{const.}$,

$$d\mathbf{S} = \hat{\boldsymbol{\phi}}\, r\, dr\, d\theta.$$

The most important case for electrostatics is $r = \text{const.}$ because Gaussian surfaces around point charges and charged spheres are spherical.

For a small patch on a sphere of radius r, a change $d\theta$ moves us along a meridian by the small distance $r\,d\theta$, while a change $d\phi$ moves us along the local circle of latitude by $r\sin\theta\,d\phi$. Their product gives

$$\boxed{dS = r^2\sin\theta\, d\theta\, d\phi}.$$



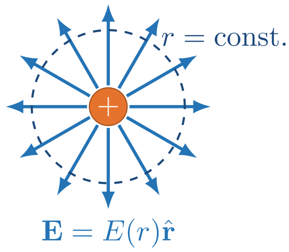
Since the outward normal to a sphere is radial,

$$\boxed{d\mathbf{S} = \hat{\mathbf{r}}\, r^2\sin\theta\, d\theta\, d\phi}.$$

18.1 Why spherical coordinates are useful

For a point charge at the origin, the system looks the same in every direction.
The electric field can therefore depend only on r and must point radially:

$$\boxed{\mathbf{E} = E(r)\hat{\mathbf{r}}}$$



A point charge has spherical symmetry.

For such a field,

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 E(r)\right).$$

If

$$E(r) \propto \frac{1}{r^2},$$

then

$$r^2 E(r) = \text{constant}.$$

Therefore,

$$\nabla \cdot \mathbf{E} = 0$$

for every point with $r \neq 0$.

But the charge is at $r = 0$.

To describe this point correctly, we need the Dirac delta function.

19. The Dirac delta function

A point charge occupies zero volume but carries finite charge.

A point source is therefore represented by a distribution rather than by an ordinary finite function.

The Dirac delta is the mathematical object used for such perfectly localized sources.

19.1 One-dimensional delta

The one-dimensional Dirac delta centered at $x = a$ is written as

$$\delta(x-a).$$

It satisfies

$$\delta(x-a) = 0, \qquad x \neq a,$$

and

$$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1.$$

Its most important property is

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)}$$

for a sufficiently regular function f.

The delta function therefore selects the value of f at one point.

19.2 The delta as a narrow-function limit

A useful way to visualize the delta function is through normalized Gaussian functions,

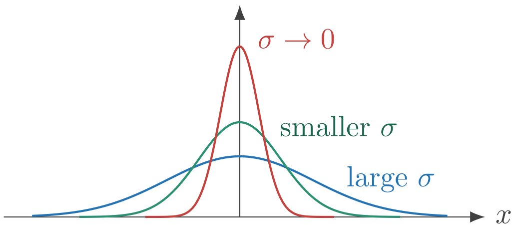
$$\delta_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

The parameter $\sigma > 0$ controls the width.

For every value of σ,

$$\int_{-\infty}^{\infty} \delta_\sigma(x)\, dx = 1.$$

As σ becomes smaller, the function becomes narrower and taller while keeping the same total area.



The width shrinks, the height grows, but the total area remains one.

In the distributional sense,

$$\delta_\sigma(x) \longrightarrow \delta(x) \qquad \text{as } \sigma \to 0.$$

19.3 Three-dimensional delta

The three-dimensional delta is written as

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}_0),$$

where $\mathbf{r}_0$ is the position of the point source.

It satisfies

$$\int f(\mathbf{r})\delta^{(3)}(\mathbf{r} - \mathbf{r}_0)\, d^3r = f(\mathbf{r}_0).$$

Here d^3r means a three-dimensional volume element.

19.4 Point charge as a density

A point charge q at $\mathbf{r}_0$ is represented by

$$\boxed{\rho(\mathbf{r}) = q\,\delta^{(3)}(\mathbf{r} - \mathbf{r}_0)}$$

The total charge is

$$Q = \int \rho(\mathbf{r})\, d^3r = q.$$

19.5 Delta function and coordinate volume

The delta function must compensate for the coordinate volume element.

In Cartesian coordinates,

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0).$$

In cylindrical coordinates,

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}_0)=\frac{\delta(\varrho-\varrho_0)\delta(\phi-\phi_0)\delta(z-z_0)}{\varrho}.$$

In spherical coordinates,

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}_0)=\frac{\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{r^2\sin\theta},$$

for a point away from coordinate singularities.

The factors in the denominator cancel the corresponding geometric factors in dV.

20. The Coulomb field and its singular source

Now consider the electric field of a point charge at the origin:

$$\mathbf{E}=\frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

For every point with $r\neq 0$,

$$\nabla\cdot\mathbf{E}=0.$$

This may look strange.

The field clearly comes from a charge, but its divergence is zero everywhere except at one point.

The missing source is located exactly at the origin.

The correct mathematical relation is

$$\boxed{\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right)=4\pi\delta^{(3)}(\mathbf{r})}$$

Therefore,

$$\boxed{\nabla\cdot\mathbf{E}=\frac{q}{\varepsilon_0}\delta^{(3)}(\mathbf{r})}$$

for a point charge at the origin.

The divergence is concentrated at one point, while the flux is finite.

20.1 Derivation

Derivation

Integrate over a sphere containing the origin:

$$\int_V \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) dV.$$

Using Gauss's theorem,

$$\int_V \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) dV = \oint_{\partial V} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{S}.$$

For a sphere of radius R,

$$d\mathbf{S} = \hat{\mathbf{r}} R^2 \sin\theta \, d\theta \, d\phi.$$

Therefore,

$$\begin{aligned} \oint_{\partial V} \frac{\hat{\mathbf{r}}}{R^2} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi \\ &= 4\pi. \end{aligned}$$

The divergence is zero away from the origin but integrates to 4π for every volume containing the origin. That is exactly the defining behavior of

$$4\pi\delta^{(3)}(\mathbf{r}).$$

Hence,

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^{(3)}(\mathbf{r}).$$

Since

$$\nabla \left(\frac{1}{r}\right) = -\frac{\hat{\mathbf{r}}}{r^2},$$

we obtain another fundamental identity:

$$\boxed{\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta^{(3)}(\mathbf{r})}$$

This relation will later connect the Coulomb potential directly to Poisson's equation.

21. How Week 1 enters Maxwell's equations

The mathematical tools introduced this week are exactly the objects that appear in Maxwell's theory.

Mathematical object	**Physical meaning**	**Electrodynamics**
$\nabla \cdot \mathbf{E}$	sources of electric field	Gauss's law
$\nabla \cdot \mathbf{B}$	sources of magnetic field	absence of magnetic monopoles
$\nabla \times \mathbf{E}$	electric circulation	Faraday induction
$\nabla \times \mathbf{B}$	magnetic circulation	Ampère–Maxwell law
$\nabla^2\Phi$	spatial structure of potential	Poisson and Laplace equations
$\int \mathbf{E} \cdot d\mathbf{l}$	field accumulated along a path	potential difference and emf
$\oint \mathbf{E} \cdot d\mathbf{S}$	electric flux through a closed surface	Gauss's law
$\delta^{(3)}(\mathbf{r})$	localized source	point charge

The same mathematical objects also come in local and global pairs:

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{E} \quad &\Longleftrightarrow \quad \oint \mathbf{E} \cdot d\mathbf{S}, \\ \nabla \times \mathbf{E} \quad &\Longleftrightarrow \quad \oint \mathbf{E} \cdot d\mathbf{l}. \end{aligned}}$$

The differential equations describe the field point by point; the integral equations describe what is measured over finite geometric objects.

21.1 Why Maxwell gives both divergence and curl

There is a deeper reason why Maxwell's equations specify both the divergence and the curl of the electromagnetic fields.

The divergence tells us about local sources and sinks. The curl tells us about local circulation. These are complementary pieces of information.

Under suitable regularity, topology, and boundary or decay conditions, the divergence and curl together determine the vector field.

Schematically,

$$\boxed{\left.\begin{matrix} \nabla \cdot \mathbf{A} \\ \nabla \times \mathbf{A} \end{matrix}\right\} + \text{boundary conditions} \quad \Longrightarrow \quad \mathbf{A}}$$

This statement is the basic idea behind the **Helmholtz decomposition**.

At the level needed here, its physical meaning is:

- divergence describes where field lines begin or end locally,
- curl describes how the field circulates locally,
- boundary conditions specify how the field behaves at the edge of the region or far away.

Maxwell's equations have exactly this structure:

$$\nabla \cdot \mathbf{E}, \qquad \nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{B}, \qquad \nabla \times \mathbf{B}.$$

Divergence and curl are therefore the natural local language of electromagnetic fields.

22. Common points to keep clear

1. The gradient acts on a scalar and produces a vector.
2. The divergence acts on a vector and produces a scalar.
3. The curl acts on a vector and produces a vector.
4. The Laplacian of a scalar is the divergence of its gradient.
5. The divergence describes local source strength, not simply whether field lines look outward globally.
6. The curl describes local circulation, not simply whether field lines look curved.

7. A line integral uses the tangent element $d\mathbf{l}$.
8. A flux integral uses the normal surface element $d\mathbf{S}$.
9. For a closed surface, $d\mathbf{S}$ points outward.
10. The unit vectors in cylindrical and spherical coordinates depend on position.
11. The factors ϱ and $r^2 \sin\theta$ in volume elements arise from geometry.
12. The Coulomb field has zero ordinary divergence for $r \neq 0$ but a delta-function source at $r = 0$.
13. A partial derivative such as $\partial\mathbf{B}/\partial t$ measures time variation at fixed position.
14. For a conservative field, the line integral between two points is path independent and the closed-loop integral is zero.
15. In cylindrical and spherical coordinates, the geometric factors in $d\mathbf{l}$, $d\mathbf{S}$, and dV must be included consistently.
16. The fundamental theorem for gradients, Stokes' theorem, and Gauss's theorem all convert local derivative information into information on a boundary.
17. Divergence, curl, and appropriate boundary conditions together contain the information needed to determine a vector field under the usual conditions.

23. Week 1 at a glance

The logical structure of this week is

$$\boxed{\begin{array}{c}
\text{position and vectors} \to \text{scalar and vector fields} \to \text{space and time dependence} \\
\Downarrow \\
\nabla \to \text{gradient} \quad \text{divergence} \quad \text{curl} \quad \text{Laplacian} \\
\Downarrow \\
\text{line integrals} \to \text{conservative fields} \to \text{surface and volume integrals} \\
\Downarrow \\
\text{gradient theorem} \qquad \text{Gauss theorem} \qquad \text{Stokes theorem} \\
\Downarrow \\
\text{local–global boundary structure} \to \text{coordinate systems} \\
\Downarrow \\
\text{Dirac delta} \to \text{point sources} \to \text{Maxwell-field structure}
\end{array}}$$

The central relations are

$$\boxed{\mathbf{E} = -\nabla\Phi}$$

$$\boxed{\nabla \times (\nabla f) = 0, \qquad \nabla \cdot (\nabla \times \mathbf{A}) = 0}$$

$$\boxed{\int_V \nabla \cdot \mathbf{A}\, dV = \oint_{\partial V} \mathbf{A} \cdot d\mathbf{S}}$$

$$\boxed{\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{l}}$$

and

$$\boxed{\nabla^2 \left(\frac{1}{r} \right) = -4\pi \delta^{(3)}(\mathbf{r})}$$

24. Bridge to Week 2

We now have the mathematical language required for electrostatics.

The next question is physical:

$$\boxed{\text{How does electric charge produce an electric field?}}$$

In Week 2, we will begin with Coulomb's law and the principle of superposition.

We will then use flux, divergence, and Gauss's theorem to move from the force between individual charges to the field equation

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.}$$

The mathematics developed in Week 1 will therefore become the physical electrodynamics of Week 2.