Electrodynamics & Relativity

Week 2: Electrostatics

Coulomb's law, Gauss's Law, and Potential

Abbas Ali Saberi

School of Science, Constructor University, Bremen, Germany

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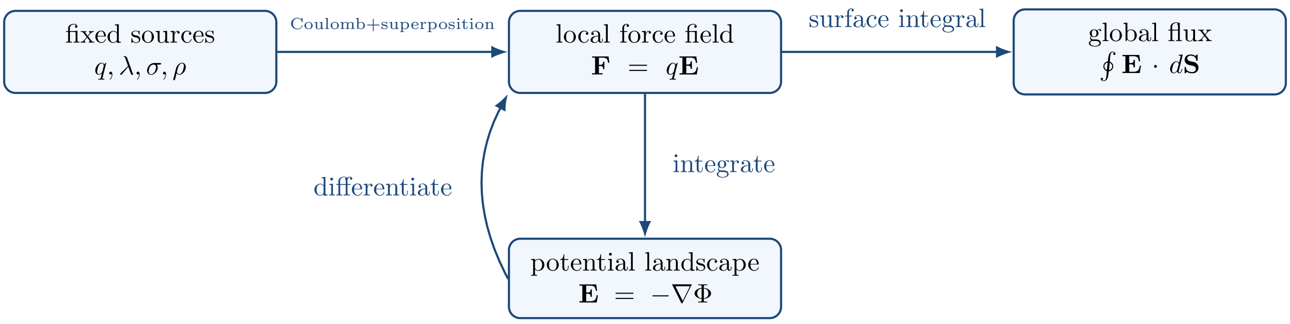
Main idea. Electrostatics asks one question: how does a fixed distribution of charge determine force, energy, and motion? We answer it in a continuous chain. Coulomb's law gives the field of one point charge. Superposition builds the field of many charges. Gauss's law reads the same field through its flux. The potential stores the same information in one scalar function.

$$\boxed{\rho \xrightarrow{\text{Coulomb}} \mathbf{E} \underset{-\nabla}{\overset{-\int \mathbf{E}\cdot d\mathbf{l}}{\longleftrightarrow}} \Phi}$$

The formulas will matter, but the main skill is to choose the right description and to test every answer by symmetry, dimensions, and limiting cases.

The week follows one connected chain of ideas:



Each arrow is a useful change of viewpoint. Gauss's law then connects the global flux directly to the enclosed charge, $\oint \mathbf{E} \cdot d\mathbf{S} = Q_{\text{enc}}/\varepsilon_0$. The worked examples below will repeatedly move around this diagram.

1. The electrostatic regime

Electrostatics describes configurations in which the electric sources are fixed in time and there is no time-dependent electromagnetic induction. We therefore study charge densities and electric fields of the form

$$\rho = \rho(\mathbf{r}), \qquad \mathbf{E} = \mathbf{E}(\mathbf{r}),$$

so that

$$\frac{\partial \rho}{\partial t} = 0, \qquad \frac{\partial \mathbf{E}}{\partial t} = 0.$$

Here $\mathbf{r}$ is the observation position and t is time. In the electrostatic regime, both the source configuration and the electric field are stationary in time. The field may still vary strongly from point to point in space.

Physical meaning

For a fixed point charge, $E \propto 1/r^2$. Two observation points at different distances can therefore measure very different field strengths, even though the field at each fixed point is constant in time.

2. Electric charge and charge distributions

Electric charge is the source of the electrostatic field. The SI unit of charge is the coulomb, written C.

We use q for a discrete charge and Q for a total charge. Charge may be concentrated at points or distributed along lines, surfaces, or volumes.

2.1 Point charges

An ideal point charge q_i located at position $\mathbf{r}_i$ can be represented by the three-dimensional Dirac delta function:

$$\rho(\mathbf{r}) = q_i\,\delta^{(3)}(\mathbf{r} - \mathbf{r}_i).$$

Here $\delta^{(3)}$ is the three-dimensional Dirac delta introduced in Week 1.

For several point charges,

$$\rho(\mathbf{r}) = \sum_i q_i\,\delta^{(3)}(\mathbf{r} - \mathbf{r}_i).$$

2.2 Continuous charge distributions

Three charge densities occur frequently.

Distribution	**Density**	**SI unit**	**Charge element**
Line charge	λ	C/m	$dq = \lambda\,dl'$
Surface charge	σ	$\mathrm{C/m^2}$	$dq = \sigma\,dS'$
Volume charge	ρ	$\mathrm{C/m^3}$	$dq = \rho\,dV'$

Here:

- λ is line charge density,
- σ is surface charge density,
- ρ is volume charge density,
- dl', dS', and dV' are source line, surface, and volume elements.

The prime reminds us that the integration variable refers to the **source position $\mathbf{r}'$**.

The total charge is therefore

$$Q = \int_L \lambda(\mathbf{r}')\,dl',$$

or

$$Q = \int_S \sigma(\mathbf{r}')\,dS',$$

or

$$Q = \int_V \rho(\mathbf{r}')\,dV'.$$

How to read the theory

Charge is integrated over the geometric object on which it is distributed. The corresponding density always has units of charge divided by the appropriate measure:

$$\lambda = \frac{dq}{dl}, \qquad \sigma = \frac{dq}{dS}, \qquad \rho = \frac{dq}{dV}.$$

This geometric bookkeeping is important because the source variable $\mathbf{r}'$ will later be integrated over a line, surface, or volume, while the observation point $\mathbf{r}$ remains fixed.

Worked example: total charge in a nonuniform sphere

Question. A solid sphere of radius R has the spherically symmetric density

$$\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right), \qquad 0 \le r \le R.$$

The density is largest at the center and falls smoothly to zero at the surface. Find the total charge.
Solution. A thin spherical layer of radius r and thickness dr has volume $dV = 4\pi r^2 dr$. Therefore,

$$\begin{aligned} Q &= 4\pi \int_0^R \rho(r) r^2\, dr \\ &= 4\pi\rho_0 \int_0^R \left(r^2 - \frac{r^3}{R}\right) dr = \boxed{\frac{\pi}{3}\rho_0 R^3}. \end{aligned}$$

The result has units $(\mathrm{C/m^3})\mathrm{m}^3 = \mathrm{C}$ ✓. It is also smaller than $4\pi\rho_0 R^3/3$ ✓, the charge of a sphere filled everywhere with the maximum density ρ_0. Both checks support the result.

3. Coulomb's law

Coulomb's law gives the electrostatic force between two point charges.

Let q_1 be at position $\mathbf{r}_1$ and q_2 at position $\mathbf{r}_2$. Define the displacement from q_1 to q_2 by

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1,$$

its magnitude by $R_{12} = |\mathbf{R}_{12}|$, and the corresponding unit vector by $\hat{\mathbf{R}}_{12} = \frac{\mathbf{R}_{12}}{R_{12}}$. The force on q_2 due to q_1 is

$$\boxed{\mathbf{F}_{1\to 2} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{R_{12}^2} \hat{\mathbf{R}}_{12}}$$

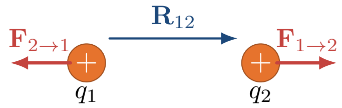
where ε_0 is the vacuum permittivity. It is often convenient to define Coulomb's constant

$$k_e = \frac{1}{4\pi\varepsilon_0} \approx 8.99 \times 10^9\ \mathrm{N\,m^2/C^2}.$$

Physical meaning

If $q_1 q_2 > 0$, the charges have the same sign and repel.
If $q_1 q_2 < 0$, the charges have opposite signs and attract.
The magnitude of the force decreases as $1/R_{12}^2$.
Newton's third law is satisfied:

$$\mathbf{F}_{2\to 1} = -\mathbf{F}_{1\to 2}.$$



Like charges repel. The force acts along the line joining the charges.

How to read the theory

Coulomb's law contains two pieces of information at once:

$$\boxed{\text{magnitude } \propto \frac{1}{R^2} \qquad \text{and} \qquad \text{direction along } \hat{\mathbf{R}}}$$

The inverse-square factor is closely connected to three-dimensional geometry: the area of a sphere grows as $4\pi R^2$. A radial field whose total outward flux is conserved therefore naturally decreases as $1/R^2$. We will make this connection exact with Gauss's law.

4. Electric field

The electric field separates the effect of the source charges from the particular charge used to probe the field. Place *a small positive test charge* q_0 at position $\mathbf{r}$. If the electrostatic force on it is $\mathbf{F}$, the electric field is the force per unit positive test charge in the limit that the test charge does not disturb the sources:

$$\boxed{\mathbf{E}(\mathbf{r}) = \lim_{q_0 \to 0} \frac{\mathbf{F}}{q_0}}$$

Once $\mathbf{E}$ is known, the electrostatic force on any charge q placed at that point is

$$\boxed{\mathbf{F} = q\mathbf{E}}.$$

For $q > 0$, the force is parallel to $\mathbf{E}$; for $q < 0$, it is opposite to $\mathbf{E}$. The SI unit of electric field is N/C, which is equivalent to V/m.

4.1 Field of a point charge

For a source charge q at position $\mathbf{r}'$, define

$$\mathbf{R} = \mathbf{r} - \mathbf{r}', \qquad R = |\mathbf{R}|, \qquad \hat{\mathbf{R}} = \frac{\mathbf{R}}{R}.$$

Then the field at observation point $\mathbf{r}$ is

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^2} \hat{\mathbf{R}}}$$

or equivalently

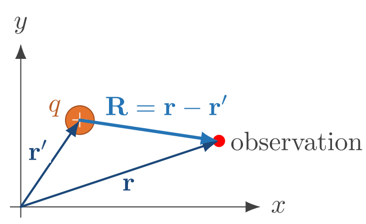
$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} q \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}}.$$

These formulas apply at observation points $\mathbf{r} \neq \mathbf{r}'$. At the ideal point-charge position itself, the field is singular.

The distinction between $\mathbf{r}'$ and $\mathbf{r}$ is essential.
$\mathbf{r}'$ tells us **where the source charge is**.
$\mathbf{r}$ tells us **where the field is measured**.
The vector $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ always points from the source to the observation point.

$$R = |\mathbf{r} - \mathbf{r}'|$$

is the physical distance between them.



Source position, observation position, and separation vector.

Worked example: where can two unequal charges give zero field?

A charge q is at $x = 0$ and a charge $4q$ is at $x = d$, with $q > 0$. Find every point on the x axis where $\mathbf{E} = 0$.

Prediction. Outside the two charges, both fields point mainly in the same direction, so they cannot cancel. Between the charges, the fields point in opposite directions. The zero must be closer to the smaller charge.

Solution. Let the point be at x, where $0 < x < d$. Cancellation requires equal magnitudes:

$$\frac{1}{4\pi\varepsilon_0}\frac{q}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{4q}{(d-x)^2}.$$

All quantities are positive in this interval, so taking the positive square root gives

$$\frac{1}{x} = \frac{2}{d-x} \qquad \Longrightarrow \qquad \boxed{x = \frac{d}{3}}.$$

The answer is closer to q, exactly as predicted. Outside the interval, the two fields point in the same direction, so no further zero exists.

Checks and physical lesson

Along the x axis, a small displacement from $d/3$ produces a restoring force on a positive test charge. But a small displacement perpendicular to the axis produces a force away from the axis. The zero-field point is therefore a *saddle*, not a stable electrostatic trap. This is a first glimpse of a general restriction that will follow from Laplace's equation.

5. Superposition principle

Electrostatics is linear: the electric field produced by several sources is the vector sum of the fields produced by the sources separately.

5.1 Discrete point charges

For point charges q_i at positions $\mathbf{r}_i$,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}}.$$

Each term is a vector and must be added as a vector.

Example

Two equal positive charges q are placed symmetrically at $x = \pm a$. At the midpoint $x = 0$, the field from the left charge points to the right and the field from the right charge points to the left. Their magnitudes are equal, so

$$\mathbf{E}(0) = 0.$$

This cancellation follows from symmetry ✓ before any detailed calculation is needed.

5.2 Continuous charge distributions

For a small source element dq at $\mathbf{r}'$, the field contribution at $\mathbf{r}$ is

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}\hat{\mathbf{R}},$$

where

$$\mathbf{R} = \mathbf{r} - \mathbf{r}'.$$

Integrating all source elements gives the total field.

For a line charge,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \lambda(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dl'}.$$

For a surface charge,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \sigma(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dS'}.$$

For a volume charge,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dV'}.$$

Derivation

For a volume distribution, divide the source into small volume elements dV'. The charge inside one element is

$$dq = \rho(\mathbf{r}')\,dV'.$$

Its contribution to the field is

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\rho(\mathbf{r}')\,dV'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}.$$

The total field is the sum of all infinitesimal contributions, so the sum becomes an integral:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dV'.$$

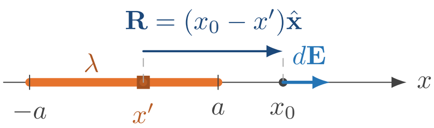
5.3 Worked example: field on the axis of a finite charged rod

A direct Coulomb integral is useful when the source has a simple geometry but not enough symmetry for Gauss's law to determine the field immediately.

Consider a uniformly charged rod on the x axis from $x' = -a$ to $x' = a$. Its line charge density is constant, λ, so its total charge is $Q = 2a\lambda$. We find the field at an observation point $x_0 > a$ on the same axis.

For a source element at position x', the charge is $dq = \lambda\,dx'$. Because the observation point lies to the right of the entire rod, every contribution points along $+\hat{\mathbf{x}}$ and $R = x_0 - x'$. Therefore,

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx'}{(x_0 - x')^2}\hat{\mathbf{x}}.$$



x' runs from $-a$ to a; $x_0 > a$ is fixed.

Integrating over the rod,

$$\mathbf{E}(x_0) = \frac{\lambda}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{dx'}{(x_0 - x')^2}\hat{\mathbf{x}} = \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{x_0 - a} - \frac{1}{x_0 + a}\right)\hat{\mathbf{x}}.$$

Thus

$$\boxed{\mathbf{E}(x_0) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x_0^2 - a^2}\hat{\mathbf{x}}, \qquad x_0 > a.}$$

Far from the rod, $x_0 \gg a$, this becomes

$$\mathbf{E}(x_0) \simeq \frac{1}{4\pi\varepsilon_0}\frac{Q}{x_0^2}\hat{\mathbf{x}},$$

so a distant finite source looks like a point charge carrying the same total charge Q.

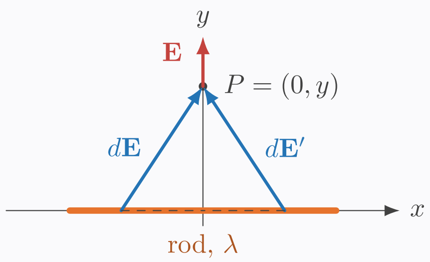
Checks and physical lesson

The exact result passes three useful tests.

- **Units:** $Q/(\varepsilon_0 x_0^2)$ has units of electric field. ✓
- **Far away:** when $x_0 \gg a$, the size of the rod is invisible and the point-charge result is recovered. ✓
- **Near an ideal endpoint:** as $x_0 \to a^+$, the field diverges. This is a property of the ideal zero-radius line model; a real rod with finite thickness removes the divergence. ✓

Worked example: field on the perpendicular bisector of a finite rod

Use the same uniformly charged rod, from $x' = -a$ to $x' = a$, but place the observation point at $(0, y)$ with $y > 0$. This geometry looks harder because each $d\mathbf{E}$ points in a different direction.



Opposite horizontal components cancel; vertical components add.

For the element $dq = \lambda\, dx'$ at x', the vertical component is

$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{\lambda y\, dx'}{(x'^2 + y^2)^{3/2}}.$$

The horizontal part is odd in x' and integrates to zero. The vertical part is even, so

$$E_y = 2\frac{1}{4\pi\varepsilon_0}\lambda y \int_0^a \frac{dx'}{(x'^2 + y^2)^{3/2}} = 2\frac{1}{4\pi\varepsilon_0}\lambda y \left[\frac{x'}{y^2\sqrt{x'^2 + y^2}}\right]_0^a.$$

Hence

$$\boxed{\mathbf{E}(0, y) = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda a}{y\sqrt{a^2 + y^2}}\,\hat{\mathbf{y}}}.$$

This one formula contains two different physical regimes:

$$\mathbf{E} \simeq \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{y^2}\hat{\mathbf{y}}, & y \gg a, \\ \dfrac{\lambda}{2\pi\varepsilon_0 y}\hat{\mathbf{y}}, & a \gg y. \end{cases}$$

Far away, the finite rod looks like a point charge ✓. Very near the middle of a very long rod, it looks like an infinite line charge ✓. The calculation has connected the $1/r^2$ and $1/r$ laws without changing the source.

6. Electric field lines

Field lines are a geometric visualization of a vector field. Their *tangent* gives the field *direction*, and their drawn *density* can be used to indicate relative field *strength*.

At each point, the tangent to a field line gives the direction of $\mathbf{E}$. A denser set of field lines is usually drawn where the field magnitude is larger.

Physical meaning

Electric field lines begin on positive charge and end on negative charge, or extend to infinity if no opposite charge is available. Two field lines *cannot cross*, because the electric field at one regular point cannot have two different directions.

7. Electric flux

Electric flux measures how much electric field passes through a surface. For a surface S,

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{S},$$

where $d\mathbf{S} = \hat{\mathbf{n}}\, dS$ is the directed surface element, dS is its area, and $\hat{\mathbf{n}}$ is the chosen unit normal. For a *closed surface*, the normal is chosen *outward* and

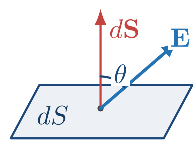
$$\boxed{\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{S}}.$$

Physical meaning

For one small surface element,

$$\boxed{\mathbf{E} \cdot d\mathbf{S} = E\, dS \cos\theta}$$

where θ is the angle between $\mathbf{E}$ and $\hat{\mathbf{n}}$. Only the component $E_\perp = E\cos\theta$ normal to the surface contributes to the flux.



Flux counts the component of $\mathbf{E}$ that crosses the surface.

The SI unit of electric flux is $\mathrm{N\,m^2/C}$.

8. Gauss's law

Gauss's law relates the total electric flux through a closed surface to the total charge enclosed by that surface.

Key relations

For any closed surface S,

$$\boxed{\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\varepsilon_0}}$$

where Q_{enc} is the total charge inside the closed surface.

A closed surface used with Gauss's law is often called a **Gaussian surface**. It is an imaginary mathematical surface, not a physical object.

8.1 Point charge, solid angle, and spherical flux

Consider a point charge q at the origin. Its electric field is radial,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}}.$$

The factor $1/r^2$ has a simple geometric meaning once we introduce **solid angle**.

Solid angle

An ordinary angle measures a fraction of a circle. A **solid angle** measures a fraction of all spatial directions around a point.

For a small surface element dS at distance r, let α be the angle between its normal $\hat{\mathbf{n}}$ and the radial direction $\hat{\mathbf{r}}$. With $d\mathbf{S} = \hat{\mathbf{n}}\,dS$,

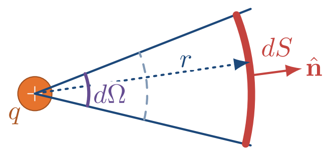
$$\boxed{d\Omega = \frac{\hat{\mathbf{r}}\cdot d\mathbf{S}}{r^2} = \frac{dS\cos\alpha}{r^2} = \frac{dS_\perp}{r^2}},$$

where $dS_\perp = dS\cos\alpha$ is the projected area perpendicular to $\hat{\mathbf{r}}$. The unit of solid angle is the **steradian**, sr.

The solid angle therefore measures the apparent size of a surface as seen from the source.
For a surface element on a sphere centered on the charge, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and $\alpha = 0$, so

$$\boxed{dS = r^2 d\Omega.}$$

Thus a fixed solid angle corresponds to an area that grows as r^2.
A useful picture is to place a unit sphere around the source: the numerical value of Ω equals the area cut out on that unit sphere.



A two-dimensional cross section of a small solid angle.

For spherical coordinates, $dS = r^2\sin\theta\,d\theta\,d\phi$,

$$\boxed{d\Omega = \sin\theta\,d\theta\,d\phi}.$$

Hence the total solid angle around a point is

$$\begin{aligned}\Omega_{\text{sphere}} &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \\ &= \boxed{4\pi\ \text{sr}}.\end{aligned}$$

Thus the 4π in Coulomb's law is directly tied to the total solid angle surrounding a point in three-dimensional space.

Electric flux per unit solid angle

For a small surface element,

$$d\Phi_E = \mathbf{E} \cdot d\mathbf{S}.$$

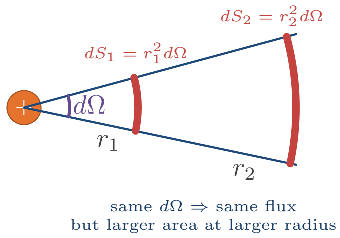
Using the Coulomb field,

$$d\Phi_E = \frac{q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}} \cdot d\mathbf{S}}{r^2} = \boxed{\frac{q}{4\pi\varepsilon_0}\, d\Omega}.$$

Therefore,

$$\boxed{\frac{d\Phi_E}{d\Omega} = \frac{q}{4\pi\varepsilon_0}}.$$

A point charge sends the same electric flux into every equal solid angle.



For a fixed $d\Omega$, $dS = r^2 d\Omega$. The same flux therefore spreads over an area growing as r^2, which gives

$$\boxed{E \propto \frac{1}{r^2}.}$$

The inverse-square field and the r^2 growth of spherical area are the same geometric statement viewed in two ways.

Flux through a complete sphere

For a sphere centered on the charge, $d\mathbf{S} = \hat{\mathbf{r}}\, R^2 d\Omega$. Hence

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{4\pi\varepsilon_0} \oint d\Omega = \frac{q}{4\pi\varepsilon_0}(4\pi) = \boxed{\frac{q}{\varepsilon_0}}.$$

The cancellation of distance is now geometric:

$$\boxed{\frac{1}{R^2} R^2 d\Omega = d\Omega.}$$

So the flux carried by a fixed solid angle is independent of how far the surface is from the charge.

How to read the theory

For a point charge,

$$\boxed{d\Phi_E = \frac{q}{4\pi\varepsilon_0}\, d\Omega.}$$

This one relation contains the main geometry: equal solid angles carry equal flux, $dS \propto r^2$ gives $E \propto 1/r^2$, and $\oint d\Omega = 4\pi$ gives $\Phi_E = q/\varepsilon_0$.

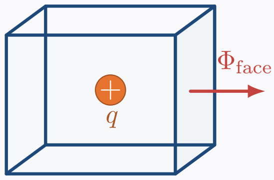
For an arbitrary tilted surface element,

$$\boxed{d\Omega = \frac{\hat{\mathbf{r}} \cdot d\mathbf{S}}{r^2}}$$

is the signed solid angle. Only the projected area contributes. The sphere is therefore not required by Gauss's law; it simply makes the solid-angle geometry maximally transparent.

Worked example: flux through one face of a cube

A point charge q is placed at the center of a cube. Find the electric flux through one face.



Gauss's law gives the flux through the whole closed cube:

$$\Phi_{\text{total}} = \frac{q}{\varepsilon_0}.$$

The centered charge makes all six faces equivalent. Therefore each face carries the same fraction of the total flux:

$$\boxed{\Phi_{\text{face}} = \frac{q}{6\varepsilon_0}}.$$

No surface integral was needed. Gauss's law fixed the total, and symmetry divided it equally. If the charge is moved away from the center but remains inside, the total flux is still q/ε_0. However, the six face fluxes are no longer equal. Gauss's law alone cannot determine each one.

Worked example: what does Coulomb's law become in pure d dimensions?

Question. Suppose both charge and the electric field live in a genuinely d-dimensional space. Assume the natural d-dimensional Gauss law $\oint \mathbf{E} \cdot d\mathbf{S} = Q_{\text{enc}}/\varepsilon_d$, where ε_d is the dimension-dependent electric coupling. Find the field and potential of a point charge q.

Solution. Rotational symmetry gives $\mathbf{E} = E(r)\hat{\mathbf{r}}$. The boundary of a d-dimensional ball is a $(d-1)$-sphere with measure $\Omega_{d-1} r^{d-1}$, where $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$. Therefore,

$$E(r)\,\Omega_{d-1} r^{d-1} = \frac{q}{\varepsilon_d} \quad \Longrightarrow \quad \boxed{\mathbf{E}(r) = \frac{q}{\varepsilon_d \Omega_{d-1}} \frac{\hat{\mathbf{r}}}{r^{d-1}}}.$$

The power law is therefore geometric: the same flux is spread over a boundary whose size grows as r^{d-1}. Using $E_r = -d\Phi/dr$ gives, for $d > 2$ and $\Phi(\infty) = 0$,

$$\boxed{\Phi(r) = \frac{q}{(d-2)\varepsilon_d \Omega_{d-1}} \frac{1}{r^{d-2}}}.$$

The special low-dimensional cases are

d	$E(r)$	$\Phi(r)$
1	constant	$\propto -\lvert x\rvert$
2	$\propto 1/r$	$-\dfrac{q}{2\pi\varepsilon_2} \ln \dfrac{r}{r_0}$
3	$\propto 1/r^2$	$\propto 1/r$

Checks and physical lesson

This is *pure* d-dimensional electrostatics. Charges confined to a two-dimensional material while the electromagnetic field still lives in ordinary three-dimensional space generally retain the three-dimensional Coulomb kernel rather than the logarithmic $2D$ one.

8.2 Differential form of Gauss's law

Gauss's law can also be written as a local differential equation.

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}}.$$

This says that electric charge density is the local source of the electric field.

Derivation

Start from the integral law and apply Gauss's divergence theorem:

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dV,$$
$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{E} \, dV.$$

Therefore,

$$\int_V \left(\nabla \cdot \mathbf{E} - \frac{\rho}{\varepsilon_0} \right) dV = 0.$$

Because this must hold for every volume V, the integrand must vanish:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

8.3 Direct connection to Coulomb's law

The differential Gauss law also follows directly from the Coulomb field integral.

For a volume charge distribution,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, dV'.$$

Here ∇ differentiates with respect to the observation coordinate $\mathbf{r}$; the source coordinate $\mathbf{r}'$ is held fixed.

Using the Week 1 identity

$$\nabla \cdot \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) = 4\pi \, \delta^{(3)}(\mathbf{r} - \mathbf{r}'),$$

we obtain

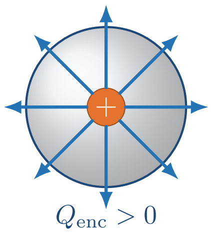
$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0} \int \rho(\mathbf{r}') \delta^{(3)}(\mathbf{r} - \mathbf{r}') \, dV' = \frac{\rho(\mathbf{r})}{\varepsilon_0}.$$

This shows that Coulomb's law and Gauss's law are two forms of the same electrostatic physics.

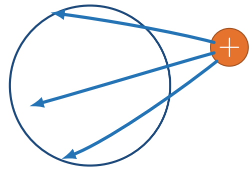
8.4 Gauss's law as a local–global statement

Gauss's law is the electrostatic version of the local–global structure introduced in Week 1. The local source density ρ inside a volume is connected to the total electric flux through its closed boundary.

$$\boxed{\underbrace{\nabla \cdot \mathbf{E} = \rho/\varepsilon_0}_{\text{local}} \iff \underbrace{\oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = Q_{\text{enc}}/\varepsilon_0}_{\text{global}}}$$



$Q_{\text{enc}} > 0$
net outward flux



$Q_{\text{enc}} = 0$
field enters and leaves;
net flux is zero

Physical meaning

A charge outside a Gaussian surface can produce a nonzero electric field on the surface. However, it contributes *zero net flux* through that closed surface: field that enters the volume also leaves it. Gauss's law therefore measures enclosed source, not the local field at one point.

9. When is Gauss's law useful?

Gauss's law is always true, but it is not always useful for calculating **E**.

It becomes especially powerful when symmetry tells us enough about the direction and magnitude of the field that **E** can be taken outside the flux integral.

The three standard symmetries are:

Symmetry	Natural Gaussian surface	Typical source
Spherical	sphere	point charge, spherical shell, charged sphere
Cylindrical	coaxial cylinder	infinite line charge
Planar	pillbox	infinite charged plane

A useful strategy is:

1. Identify the symmetry of the source.
2. Infer the direction of **E** and which coordinates its magnitude may depend on.
3. Choose a Gaussian surface adapted to that symmetry.
4. Evaluate the electric flux through each part of the surface.
5. Calculate the enclosed charge Q_{enc}.
6. Apply Gauss's law and solve for E.

How to read the theory

The power of Gauss's law comes from matching the **geometry of the source** to the **geometry of the boundary**. Symmetry does two jobs before the integral is evaluated: it fixes the direction of **E** and tells us where its magnitude is constant. Only then does the flux integral reduce to a simple product such as EA.

10. Spherical symmetry

For a spherically symmetric charge distribution centered at the origin,

$$\rho = \rho(r).$$

Rotational symmetry requires

$$\boxed{\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}}.$$

Choose a spherical Gaussian surface of radius r. On this surface, $E(r)$ is constant and $\mathbf{E}$ is parallel to $d\mathbf{S}$. Thus

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = E(r)\,4\pi r^2.$$

Gauss's law gives the general spherical result

$$\boxed{E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q_{\text{enc}}(r)}{r^2}}.$$

Here $Q_{\text{enc}}(r)$ is the charge enclosed inside radius r.

Worked example: a field that reaches its maximum inside the charge cloud

Return to the nonuniform sphere from Section 2,

$$\rho(r) = \rho_0\left(1 - \frac{r}{R}\right), \qquad 0 \le r \le R.$$

For $r < R$, the enclosed charge is

$$Q_{\text{enc}}(r) = 4\pi\rho_0 \int_0^r \left(1 - \frac{s}{R}\right) s^2\,ds = 4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^4}{4R}\right).$$

Gauss's law then gives

$$\boxed{\mathbf{E}(r) = \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4R}\right)\hat{\mathbf{r}}, \qquad r < R.}$$

Outside, the field depends only on the total charge $Q = \pi\rho_0 R^3/3$:

$$\boxed{\mathbf{E}(r) = \frac{\rho_0 R^3}{12\varepsilon_0 r^2}\hat{\mathbf{r}}, \qquad r \ge R.}$$

The field is continuous at $r = R$. More surprisingly, its maximum is not at the surface. Inside,

$$\frac{dE}{dr} = \frac{\rho_0}{\varepsilon_0}\left(\frac{1}{3} - \frac{r}{2R}\right),$$

so

$$\boxed{r_{\max} = \frac{2R}{3}}, \qquad \boxed{E_{\max} = \frac{\rho_0 R}{9\varepsilon_0}}.$$

Beyond $2R/3$, the enclosed charge still grows, but the Gaussian area $4\pi r^2$ grows fast enough to make the field decrease. This is why "more enclosed charge" does not always mean "larger field."

10.1 Thin spherical shell

Consider a thin spherical shell of radius R carrying total charge Q uniformly over its surface.

For $r > R$, the Gaussian sphere encloses all the charge: $Q_{\text{enc}} = Q$.

Therefore,

$$\boxed{\mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}, \qquad r > R.}$$

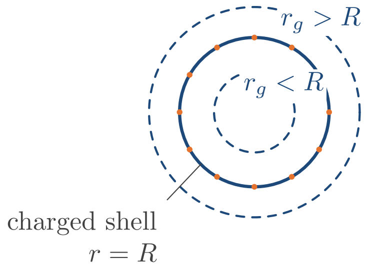
For $r < R$, the Gaussian sphere encloses no charge: $Q_{\text{enc}} = 0$,

so

$$\boxed{\mathbf{E} = 0, \qquad r < R.}$$

Physical meaning

Outside a uniformly charged spherical shell, the field is exactly the same as if the entire charge Q were concentrated at the center.
Inside the shell, the electric field is zero everywhere. This is an exact result of spherical symmetry and Gauss's law.



At the ideal surface $r = R$, the normal electric field has different one-sided limits just inside and just outside the charged shell. We will derive the general surface boundary condition in Week 3.

10.2 Uniformly charged solid sphere

Now consider a solid sphere of radius R with constant volume charge density ρ_0.

The total charge is

$$Q = \rho_0 \frac{4\pi R^3}{3}.$$

For $r \geq R$,

$$\boxed{\mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}.}$$

For $r < R$, the enclosed charge is

$$Q_{\text{enc}}(r) = \rho_0 \frac{4\pi r^3}{3}.$$

Gauss's law gives

$$E(r) 4\pi r^2 = \frac{1}{\varepsilon_0}\rho_0 \frac{4\pi r^3}{3},$$

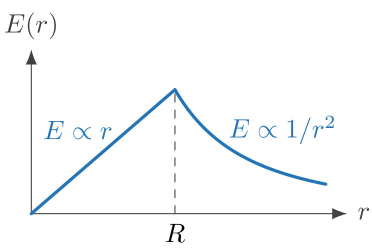
so

$$\boxed{\mathbf{E}(r) = \frac{\rho_0}{3\varepsilon_0} r\,\hat{\mathbf{r}}, \qquad r < R.}$$

The field grows linearly from zero at the center, reaches its maximum at $r = R$, and then decreases as $1/r^2$ outside.
At the surface the two expressions agree:

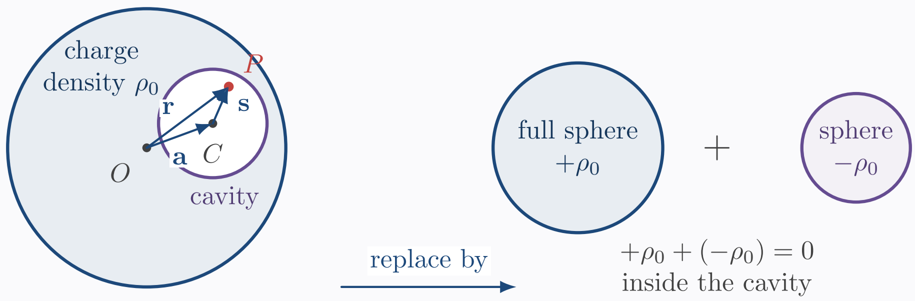
$$E(R) = \frac{\rho_0 R}{3\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}.$$

Thus $E(r)$ is continuous for a uniform volume charge density.



Worked example: an off-center cavity with a uniform field

A solid sphere has uniform charge density ρ_0. A smaller spherical cavity is cut out of it. The cavity center is displaced from the large-sphere center by the vector $\mathbf{a}$. Find the electric field everywhere inside the cavity.



actual geometry **equivalent superposition**

For a point P inside the cavity, $\mathbf{a}$ points from O to C, $\mathbf{s}$ points from C to P, and $\mathbf{r}$ points from O to P.

Key idea. Replace the cavity problem by two simpler charge distributions:

1. a complete sphere of density $+\rho_0$ centered at O;
2. a smaller sphere of density $-\rho_0$ centered at C.

Their densities add to zero inside the smaller sphere, so their superposed field is exactly the field of the cavity.
For a point P inside the cavity, define

$$\mathbf{r} = \overrightarrow{OP}, \qquad \mathbf{a} = \overrightarrow{OC}, \qquad \mathbf{s} = \overrightarrow{CP}.$$

The geometry therefore gives

$$\boxed{\mathbf{r} = \mathbf{a} + \mathbf{s}}.$$

For a uniformly charged solid sphere, the electric field at an interior point is proportional to the displacement from that sphere's center.
For the large sphere centered at O,

$$\mathbf{E}_+ = \frac{\rho_0}{3\varepsilon_0}\mathbf{r}.$$

For the fictitious sphere of density $-\rho_0$ centered at C,

$$\mathbf{E}_- = -\frac{\rho_0}{3\varepsilon_0}\mathbf{s}.$$

Therefore,

$$\begin{aligned} \mathbf{E}_{\text{cavity}} &= \mathbf{E}_+ + \mathbf{E}_- \\ &= \frac{\rho_0}{3\varepsilon_0}\left(\mathbf{r} - \mathbf{s}\right). \end{aligned}$$

Since $\mathbf{r} = \mathbf{a} + \mathbf{s}$, we have $\mathbf{r} - \mathbf{s} = \mathbf{a}$. Hence

$$\boxed{\mathbf{E}_{\text{cavity}} = \frac{\rho_0}{3\varepsilon_0}\mathbf{a}}.$$

The position P has disappeared from the answer. The field inside the off-center cavity is therefore **uniform**, even though the electric field inside the original full sphere varies with position.

Checks and physical lesson

If the cavity is concentric, then $\mathbf{a} = 0$ and the field inside is zero ✓.
If the cavity is displaced, then for $\rho_0 > 0$ the uniform field points in the direction of $\mathbf{a}$, from the large-sphere center O toward the cavity center C ✓.
The result does not depend on the position P inside the cavity and does not depend on the cavity radius, provided the cavity remains fully inside the large sphere.

11. Cylindrical symmetry: infinite line charge

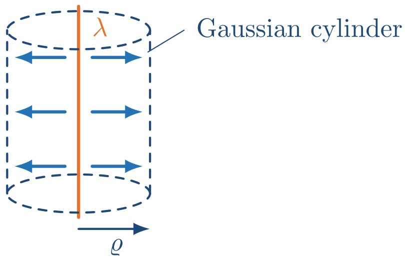
Consider an infinitely long straight line with uniform line charge density λ. Choose the line to lie along the z axis.

Cylindrical symmetry implies

$$\boxed{\mathbf{E} = E(\varrho)\hat{\boldsymbol{\varrho}}},$$

where ϱ is the perpendicular distance from the line.

Choose a cylindrical Gaussian surface of radius ϱ and length L.



The field is perpendicular to the curved side of the cylinder and has constant magnitude there.
On the two flat end caps, $\mathbf{E}$ lies parallel to the surfaces, so

$$\mathbf{E} \cdot d\mathbf{S} = 0.$$

The total flux is therefore only through the curved surface:

$$\Phi_E = E(\varrho)(2\pi\varrho L).$$

The enclosed charge is

$$Q_{\text{enc}} = \lambda L.$$

Gauss's law gives

$$E(\varrho)(2\pi\varrho L) = \frac{\lambda L}{\varepsilon_0}.$$

Therefore,

$$\boxed{\mathbf{E}(\varrho) = \frac{\lambda}{2\pi\varepsilon_0\varrho}\hat{\boldsymbol{\varrho}}}.$$

The field of an infinite line charge decreases as $1/\varrho$, not as $1/\varrho^2$.

Worked example: field inside and outside a uniformly charged cylinder

An infinitely long solid cylinder of radius R has uniform volume charge density ρ_0. Find $\mathbf{E}(\varrho)$.
Inside, $\varrho < R$. A coaxial Gaussian cylinder of length L encloses

$$Q_{\text{enc}} = \rho_0\pi\varrho^2 L.$$

Only the curved side carries flux, so

$$E(\varrho)(2\pi\varrho L) = \frac{\rho_0\pi\varrho^2 L}{\varepsilon_0}.$$

Thus

$$\boxed{\mathbf{E}(\varrho) = \frac{\rho_0\varrho}{2\varepsilon_0}\hat{\boldsymbol{\varrho}}, \qquad \varrho < R.}$$

Outside, $\varrho \geq R$. Now the Gaussian cylinder encloses the full cross-section:

$$Q_{\text{enc}} = \rho_0\pi R^2 L.$$

Therefore,

$$\boxed{\mathbf{E}(\varrho) = \frac{\rho_0 R^2}{2\varepsilon_0 \varrho}\hat{\boldsymbol{\varrho}}, \qquad \varrho \geq R.}$$

The field grows linearly inside and falls as $1/\varrho$ outside. The two formulas agree at $\varrho = R$. Far outside, the cylinder acts like a line with charge per unit length $\lambda = \rho_0 \pi R^2$.

12. Planar symmetry: infinite charged sheet

Consider an infinite plane with uniform surface charge density σ. Take the plane to be $z = 0$.

Translational symmetry in the x and y directions implies that the field cannot depend on x or y. Rotational symmetry within the plane forbids any preferred direction parallel to the plane. Therefore the field must be perpendicular to the plane.

For $\sigma > 0$,

$$\mathbf{E} = \begin{cases} E\,\hat{\mathbf{z}}, & z > 0, \\ -E\,\hat{\mathbf{z}}, & z < 0. \end{cases}$$

Choose a thin cylindrical Gaussian pillbox of area A crossing the sheet. The curved side contributes no flux because $\mathbf{E}$ is parallel to it. The two flat faces each contribute EA. Thus

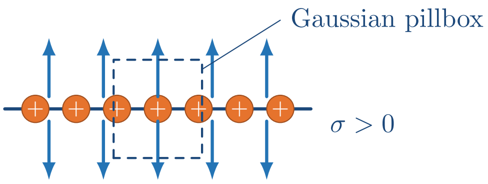
$$\Phi_E = 2EA.$$

The enclosed charge is $Q_{\text{enc}} = \sigma A$. Gauss's law gives

$$2EA = \frac{\sigma A}{\varepsilon_0},$$

so

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}.$$

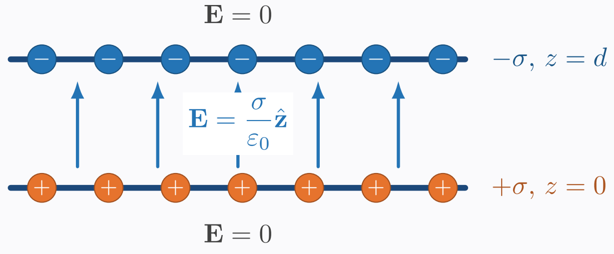


Physical meaning

The field of an ideal infinite charged plane is independent of the distance from the plane. This differs from point and line sources because the amount of source seen at larger distances grows with the geometry in exactly the way needed to keep the field magnitude constant.

Worked example: two sheets make an ideal capacitor

Place a sheet with charge density $+\sigma$ at $z = 0$ and a parallel sheet with charge density $-\sigma$ at $z = d$. Find the field in all three regions.



One positive sheet produces a field of magnitude $\sigma/(2\varepsilon_0)$ away from itself. One negative sheet produces the same magnitude toward itself. Superposition gives

$$\boxed{\mathbf{E} = \begin{cases} \mathbf{0}, & z < 0, \\ \dfrac{\sigma}{\varepsilon_0}\hat{\mathbf{z}}, & 0 < z < d, \\ \mathbf{0}, & z > d. \end{cases}}$$

Outside, the two fields cancel. Between the sheets, they add. The potential difference follows directly:

$$\boxed{\Phi(d) - \Phi(0) = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = -\frac{\sigma d}{\varepsilon_0}}.$$

The field points from the positive sheet to the negative sheet. The potential decreases in the same direction, so the negative sheet is at lower potential.

13. Electrostatic force is conservative

In electrostatics,

$$\boxed{\nabla \times \mathbf{E} = 0}.$$

Therefore the line integral of $\mathbf{E}$ between two points is path independent, and around every closed path in a simply connected electrostatic region,

$$\boxed{\oint_C \mathbf{E} \cdot d\mathbf{l} = 0}.$$

This allows us to describe the electrostatic field using a scalar potential.

14. Work and electric potential

Suppose a charge q_0 moves through an electrostatic field $\mathbf{E}$. The electric force is

$$\mathbf{F} = q_0\mathbf{E}.$$

For a small displacement $d\mathbf{l}$, the work done by the electric field is

$$dW_{\text{field}} = \mathbf{F} \cdot d\mathbf{l} = q_0\mathbf{E} \cdot d\mathbf{l}.$$

For motion from point a to point b,

$$W_{\text{field},a\to b} = q_0 \int_a^b \mathbf{E}\cdot d\mathbf{l}.$$

For a conservative force, the change in potential energy U is the negative of the work done by the field:

$$U_b - U_a = -W_{\text{field},a\to b}.$$

Define electric potential Φ as potential energy per unit charge:

$$\boxed{\Phi = \frac{U}{q_0}}.$$

Therefore,

$$\boxed{\Phi(\mathbf{r}_b) - \Phi(\mathbf{r}_a) = -\int_a^b \mathbf{E}\cdot d\mathbf{l}}.$$

The SI unit of potential is the volt:

$$1\ \text{V} = 1\ \text{J/C}.$$

14.1 Choice of zero potential

Only potential differences are physically relevant in electrostatics. We may add any constant C to the potential without changing the electric field:

$$\Phi \to \Phi + C.$$

For localized charge distributions, a convenient convention is

$$\Phi(\infty) = 0.$$

Then

$$\boxed{\Phi(\mathbf{r}) = -\int_\infty^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}}.$$

For infinite charge distributions, $\Phi(\infty)$ may not be finite. In such cases, we choose a finite reference point and work with potential differences.

Worked example: a hard-looking line integral with an easy path

Question. Two equal charges q sit at $(\pm a, 0, 0)$. Their electric field is

$$\mathbf{E}(x,y,z) = \frac{1}{4\pi\varepsilon_0} q \left[\frac{(x-a)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\left((x-a)^2 + y^2 + z^2\right)^{3/2}} + \frac{(x+a)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\left((x+a)^2 + y^2 + z^2\right)^{3/2}} \right].$$

With $\Phi(\infty) = 0$, find $\Phi(0,0,0)$ from $\Phi = -\int_\infty^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}$. Do not attempt a generic three-dimensional path.

Solution. Electrostatic line integrals are path independent, so choose the z axis: $x = y = 0$. The x components cancel and

$$\mathbf{E}(0,0,z) = \frac{1}{4\pi\varepsilon_0} \frac{2qz}{(a^2+z^2)^{3/2}}\,\hat{\mathbf{z}}.$$

Hence

$$\Phi(0) = -2k_e q \int_\infty^0 \frac{z\,dz}{(a^2+z^2)^{3/2}} = \boxed{\frac{2k_e q}{a}}.$$

The main step was not integration; it was recognizing that a clever path can turn a difficult vector integral into a one-line calculation.

Worked example: which way does an electron accelerate?

An electron starts from rest at a point where $\Phi_a = 0$ and moves to a point where $\Phi_b = +100\,\mathrm{V}$. Find its change in potential energy and its final speed.
The potential difference is

$$\Delta\Phi = \Phi_b - \Phi_a = +100\,\mathrm{V}.$$

Because the electron has charge $q = -e$,

$$\boxed{\Delta U = q\Delta\Phi = (-e)(100\,\mathrm{V}) = -100\,\mathrm{eV}}.$$

The electric field does positive work, so the loss of potential energy becomes kinetic energy:

$$\Delta K = -\Delta U = 100\,\mathrm{eV}.$$

Using $K = \frac{1}{2}m_e v^2$,

$$v = \sqrt{\frac{2(100\,\mathrm{eV})}{m_e}} \simeq \boxed{5.9 \times 10^6\,\mathrm{m/s}}.$$

This is about $0.020c$, so the nonrelativistic kinetic-energy formula is self-consistent.

Checks and physical lesson

The field points toward lower potential, but a negative charge accelerates opposite to the field. An electron therefore accelerates toward higher potential while its potential energy decreases. The sign of q is essential.

15. Potential of a point charge

For a point charge q at the origin, $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$. Choose $\Phi(\infty) = 0$. Let s denote the radial integration coordinate along the path from infinity to the observation radius r. Then

$$d\mathbf{l} = ds\,\hat{\mathbf{r}},$$

and

$$\Phi(r) = -\int_\infty^r \frac{1}{4\pi\varepsilon_0}\frac{q}{s^2}\,ds.$$

The result is

$$\boxed{\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}.$$

Derivation

Since

$$\int s^{-2}\,ds = -\frac{1}{s},$$

we find

$$\Phi(r) = -\frac{q}{4\pi\varepsilon_0}\int_\infty^r s^{-2}\,ds = -\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{s}\right]_\infty^r = \frac{q}{4\pi\varepsilon_0 r}.$$

For a positive charge, $\Phi > 0$ when the zero is chosen at infinity. For a negative charge, $\Phi < 0$.

16. Superposition for the potential

Potential is a scalar, so superposition is often easier for Φ than for $\mathbf{E}$. For point charges q_i at positions $\mathbf{r}_i$,

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}}.$$

For continuous distributions,

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dl'}$$

for a line charge,

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dS'}$$

for a surface charge, and

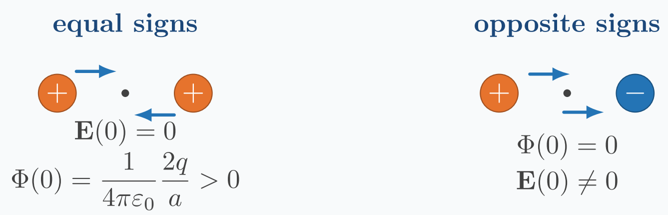
$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'}$$

for a volume charge.

These absolute-potential formulas are directly useful when the chosen reference makes the integrals finite, as for localized charge distributions with $\Phi(\infty) = 0$. For ideal infinite line or plane charges, use potential differences relative to a finite reference point instead.

Example

Field and potential can cancel differently.
Consider two charges placed symmetrically at $x = \pm a$. At the midpoint, the result depends strongly on whether the charges have equal or opposite signs.



For two equal positive charges, the vector fields cancel at the midpoint while the scalar potentials add. For a positive–negative pair, the scalar potentials cancel at the midpoint while the two electric-field contributions point in the same direction.
The field is a vector and the potential is a scalar, so their cancellations follow different rules.

16.1 Far away: total charge first, dipole next

A distant observer cannot resolve every detail of a localized charge distribution. For source size a and observation distance $r \gg a$,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r^2} + O\left(\frac{a^2}{r^3}\right).$$

Therefore the first two source quantities that survive far away are $Q = \int \rho(\mathbf{r}')\,d^3r'$ and $\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')\,d^3r'$, giving

$$\boxed{\Phi(\mathbf{r}) \simeq \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right)}.$$

If $Q = 0$ but $\mathbf{p} \neq 0$, the leading far field is the dipole field

$$\boxed{\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3}}.$$

Worked example: a neutral ring that becomes a pure dipole far away

A ring of radius a lies in the xy plane and carries the nonuniform line density

$$\lambda(\phi) = \lambda_0 \sin\phi, \qquad 0 \leq \phi < 2\pi.$$

Find the exact electric field on the z axis and identify its far-field form.
First, $dq = \lambda_0 a \sin\phi \, d\phi$, so

$$Q = \int_0^{2\pi} dq = 0.$$

The dipole moment is

$$\begin{aligned}\mathbf{p} &= \oint \mathbf{r}' \, dq \\ &= \lambda_0 a^2 \int_0^{2\pi} (\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) \sin\phi \, d\phi = \boxed{\pi\lambda_0 a^2\,\hat{\mathbf{y}}}.\end{aligned}$$

Every element is the same distance $R = \sqrt{a^2 + z^2}$ from an axial observation point, but the vector numerator matters. Direct superposition gives

$$\boxed{\mathbf{E}(0,0,z) = -\frac{1}{4\pi\varepsilon_0}\frac{\pi\lambda_0 a^2}{(a^2+z^2)^{3/2}}\,\hat{\mathbf{y}} = -\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{(a^2+z^2)^{3/2}}}.$$

For $|z| \gg a$,

$$\boxed{\mathbf{E}(0,0,z) \simeq -\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{|z|^3}}.$$

This is exactly the equatorial dipole field because $\mathbf{p} \perp \hat{\mathbf{z}}$. A useful surprise is that $\Phi = 0$ everywhere on the z axis—all source elements have the same distance and the total charge is zero—yet $\mathbf{E} \neq 0$ there because the transverse derivative of Φ is not zero.

17. Electric field from the potential

The electric field is obtained from the spatial variation of the potential:

$$\boxed{\mathbf{E} = -\nabla\Phi}.$$

In Cartesian coordinates,

$$\boxed{\mathbf{E} = -\frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} - \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} - \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}}.$$

Thus

$$E_x = -\frac{\partial\Phi}{\partial x}, \qquad E_y = -\frac{\partial\Phi}{\partial y}, \qquad E_z = -\frac{\partial\Phi}{\partial z}.$$

The minus sign means that $\mathbf{E}$ points in the direction of greatest decrease of Φ.

Worked example: a zero-field point that is not an energy minimum

In a two-dimensional region, suppose

$$\Phi(x,y) = \Phi_0 \frac{x^2 - y^2}{a^2},$$

where Φ_0 and a are constants. Find the field, locate its zero, and decide whether that point is a minimum of Φ.
Taking the gradient gives

$$\boxed{\mathbf{E}(x,y) = -\nabla\Phi = -\frac{2\Phi_0 x}{a^2}\hat{\mathbf{x}} + \frac{2\Phi_0 y}{a^2}\hat{\mathbf{y}}}.$$

The field vanishes only at the origin. However, the origin is not a minimum:

$$\Phi(x,0) = +\Phi_0\frac{x^2}{a^2}, \qquad \Phi(0,y) = -\Phi_0\frac{y^2}{a^2}.$$

For $\Phi_0 > 0$, the potential rises along the x axis but falls along the y axis. The origin is a saddle point. Also,

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = \frac{2\Phi_0}{a^2} - \frac{2\Phi_0}{a^2} = 0.$$

So this nonzero field can exist in a charge-free region. This example prepares us for Laplace's equation in Week 3: empty space does not require $\mathbf{E} = 0$; it requires the local sources to vanish.

17.1 Equipotential surfaces

An equipotential surface is a surface on which Φ is constant. Along a small displacement $d\mathbf{l}$ tangent to an equipotential,

$$d\Phi = 0.$$

But

$$d\Phi = \nabla\Phi \cdot d\mathbf{l} = -\mathbf{E}\cdot d\mathbf{l}.$$

Therefore,

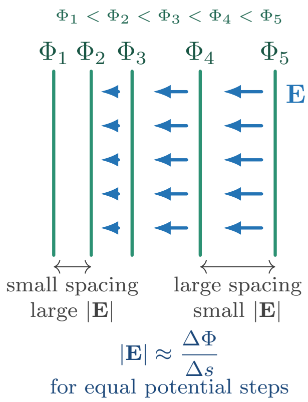
$$\mathbf{E}\cdot d\mathbf{l} = 0.$$

So the electric field is perpendicular to equipotential surfaces.

Physical meaning

Equipotential surfaces provide a geometric picture of the potential.

- $\mathbf{E}$ is perpendicular to them.
- $\mathbf{E}$ points toward lower Φ.
- Closely spaced equipotentials indicate a larger field magnitude because the potential changes more rapidly with distance.



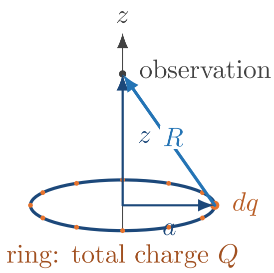
18. Example: potential on the axis of a charged ring

This example shows why the scalar potential can be easier to calculate than the vector electric field.

Consider a ring of radius a carrying total charge Q uniformly. The ring lies in the xy plane and is centered at the origin. We want the potential at an observation point on the z axis.
Every source element dq is the same distance from that observation point:

$$\boxed{R = \sqrt{a^2 + z^2}}.$$

Because R is independent of the position of dq around the ring, the potential integral becomes especially simple.
Therefore,



$$d\Phi = \frac{1}{4\pi\varepsilon_0} \frac{dq}{\sqrt{a^2 + z^2}}.$$

Because the denominator is constant around the ring,

$$\Phi(z) = \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{a^2 + z^2}} \int dq.$$

Since $\int dq = Q$,

$$\boxed{\Phi(z) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{a^2 + z^2}}}.$$

By symmetry, the electric field on the axis has only a z component, so

$$E_z = -\frac{d\Phi}{dz}.$$

Differentiating gives

$$\boxed{\mathbf{E}(z) = \frac{1}{4\pi\varepsilon_0} \frac{Qz}{(a^2 + z^2)^{3/2}} \hat{\mathbf{z}}}.$$

Derivation

The same result follows directly from vector superposition. For one charge element,

$$dE = \frac{1}{4\pi\varepsilon_0} \frac{dq}{R^2}.$$

The transverse components cancel pairwise around the ring. The surviving axial component is

$$dE_z = dE \frac{z}{R} = \frac{1}{4\pi\varepsilon_0} \frac{z\, dq}{R^3}.$$

Since $R = \sqrt{a^2 + z^2}$ is constant around the ring and $\int dq = Q$,

$$E_z = \frac{1}{4\pi\varepsilon_0} \frac{Qz}{(a^2 + z^2)^{3/2}}.$$

Physical meaning

The direct field calculation requires vector components and a symmetry cancellation. The potential calculation adds only scalars, and one derivative then gives the field. This is why potential methods are often simpler for extended charge distributions.

Worked example: where is the ring field strongest?

For $Q > 0$ and $z > 0$, the axial field is

$$E_z(z) = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(a^2+z^2)^{3/2}}.$$

It is zero at the center, and it also approaches zero far away. Therefore it must reach a maximum at an intermediate distance. Differentiate:

$$\frac{dE_z}{dz} = \frac{1}{4\pi\varepsilon_0}Q\frac{a^2-2z^2}{(a^2+z^2)^{5/2}}.$$

The derivative vanishes when

$$a^2 - 2z^2 = 0,$$

so

$$\boxed{z_{\max} = \frac{a}{\sqrt{2}}}, \qquad \boxed{E_{\max} = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{3\sqrt{3}\,a^2}}.$$

The same exact field also has two simple limits:

$$E_z \simeq \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{a^3}z, & z \ll a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{z^2}, & z \gg a. \end{cases}$$

Near the center, the field grows linearly. Far away, the ring looks like a point charge. The maximum lies where the observation distance becomes comparable with the ring radius.

19. Potential of a uniformly charged spherical shell

For a thin spherical shell of radius R and total charge Q, we already found

$$\mathbf{E}(r) = \begin{cases} 0, & r < R, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{\mathbf{r}}, & r > R. \end{cases}$$

Choose $\Phi(\infty) = 0$. For $r \geq R$,

$$\boxed{\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}}.$$

Inside the shell, $\mathbf{E} = 0$, so

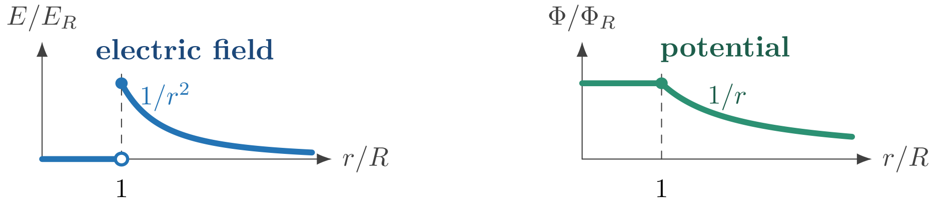
$$\nabla\Phi = 0.$$

Therefore Φ is constant throughout the interior. Its value must match the value at the surface:

$$\boxed{\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R}, \qquad r < R.}$$

Thus the field can be zero in a region even though the potential is nonzero there.

The field and potential contain the same information, but their behavior at the charged surface looks different:



Here $E_R = \dfrac{1}{4\pi\varepsilon_0}Q/R^2$ and $\Phi_R = \dfrac{1}{4\pi\varepsilon_0}Q/R$.

The potential is continuous at $r = R$, while its radial slope changes suddenly. Because $E_r = -d\Phi/dr$, this change of slope is the jump in the normal electric field caused by the surface charge. The open and filled dots in the field plot show the inside and outside limits at the ideal surface.

Worked example: oppositely charged hemispheres: the far field without solving the full field

A spherical surface of radius R carries $+\sigma_0$ on the northern hemisphere and $-\sigma_0$ on the southern hemisphere:

$$\sigma(\theta) = \begin{cases} +\sigma_0, & 0 \le \theta < \pi/2, \\ -\sigma_0, & \pi/2 < \theta \le \pi. \end{cases}$$

Find the leading electric field far from the sphere. The total charge is zero, so the monopole term vanishes. By axial symmetry $\mathbf{p} = p\hat{\mathbf{z}}$, with

$$\begin{aligned} p &= \int R\cos\theta\,\sigma(\theta)\,R^2\sin\theta\,d\theta\,d\phi \\ &= 2\pi\sigma_0 R^3 \int_0^\pi |\cos\theta|\sin\theta\,d\theta = \boxed{2\pi\sigma_0 R^3}. \end{aligned}$$

Thus, for $r \gg R$,

$$\boxed{\Phi(r,\theta) \simeq \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}}, \qquad \boxed{\mathbf{E}(r,\theta) \simeq \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)}.$$

The detailed near field is complicated, but the distant field required only one integral: the dipole moment. This is the practical power of a multipole expansion.

20. Electrostatic potential energy

If a charge q is placed in an external electrostatic potential Φ, its potential energy is

$$\boxed{U = q\Phi}.$$

For two point charges q_1 and q_2 separated by distance R, choose zero energy at infinite separation. The potential produced by q_1 at the position of q_2 is

$$\Phi_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{R}.$$

Therefore,

$$\boxed{U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{R}}.$$

If $q_1 q_2 > 0$, then $U > 0$ relative to infinite separation. If $q_1 q_2 < 0$, then $U < 0$.

For several point charges, the total electrostatic interaction energy is

$$\boxed{U = \frac{1}{4\pi\varepsilon_0}\sum_{i<j}\frac{q_i q_j}{R_{ij}}}.$$

The condition $i < j$ ensures that every pair is counted once. This is the interaction energy of distinct ideal point charges; it does not include the divergent self-energy of an ideal point charge.

Worked example: assembling four charges on a square

Four equal positive charges q are placed at the corners of a square of side a. How much external work is needed to assemble the configuration slowly from infinite separation?
There are six distinct pairs, not four. Four pairs lie along the edges and have separation a. Two pairs lie along the diagonals and have separation $\sqrt{2}a$. Therefore,

$$\begin{aligned} U &= \frac{1}{4\pi\varepsilon_0}q^2\left(\frac{4}{a} + \frac{2}{\sqrt{2}a}\right) \\ &= \boxed{\frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\left(4 + \sqrt{2}\right)}. \end{aligned}$$

For slow assembly, the change in kinetic energy is negligible, so the external work is

$$W_{\text{ext}} = \Delta U = U > 0.$$

Positive work is required because every new charge must be pushed against repulsion from the charges already present. Counting geometrically different pairs is safer than trying to write the sum from memory.

Worked example: Coulomb gas: why screening restores extensivity

Consider N identical charges q confined in a large spherical region of radius R. The number density is kept fixed,

$$n = \frac{N}{(4\pi/3)R^3},$$

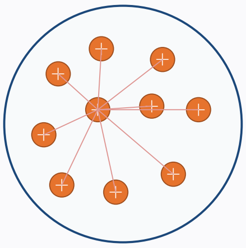
so increasing N also increases the size of the system:

$$\boxed{R = \left(\frac{3N}{4\pi n}\right)^{1/3} \propto N^{1/3}.}$$

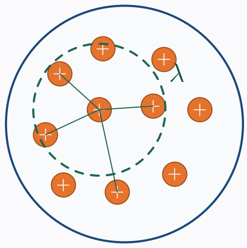
We ask a simple thermodynamic question:

$$\boxed{\text{Does the interaction energy grow as } N \text{ or faster than } N?}$$

An extensive system has $U \propto N$, so that the energy per particle U/N remains finite as $N \to \infty$.



one charge feels particles throughout the whole sample



only a finite neighborhood of size λ matters

Long-range Coulomb interactions connect a particle to the whole system. Screening cuts the interaction off beyond a finite length λ.

1. Unscreened Coulomb interaction.
To obtain the scaling cleanly, replace the discrete charges by a uniformly charged sphere with the same total charge

$$Q = Nq.$$

Its uniform charge density is

$$\rho_q = \frac{Q}{(4\pi/3)R^3}.$$

We can build the sphere shell by shell. When the sphere has already been built up to radius r, the charge inside is

$$Q(r) = Q\frac{r^3}{R^3}.$$

The charge added in the thin shell from r to $r + dr$ is

$$dQ = 4\pi r^2 \rho_q \, dr = 3Q\frac{r^2}{R^3}\, dr.$$

Before adding this shell, the potential at radius r produced by the charge already inside is

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r}.$$

Therefore the work needed to add the shell is

$$dU = \Phi(r)\, dQ = \frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r}\, dQ.$$

Substituting $Q(r)$ and dQ,

$$dU = 3\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R^6} r^4\, dr.$$

Hence

$$\begin{aligned} U_{\mathrm{C}} &= 3\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R^6}\int_0^R r^4\, dr \\ &= \boxed{\frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R}}. \end{aligned}$$

Now use $Q = Nq$ and, at fixed density, $R \propto N^{1/3}$:

$$U_{\mathrm{C}} \propto \frac{N^2}{N^{1/3}} = \boxed{N^{5/3}}.$$

Therefore,

$$\boxed{\frac{U_{\mathrm{C}}}{N} \propto N^{2/3}}.$$

The energy per particle grows without bound as the system becomes larger. The unscreened Coulomb interaction is therefore **non-extensive** for a system carrying a net charge.

Physical meaning

The origin of the problem is easy to see. The Coulomb interaction decreases only as $1/r$. As the sample becomes larger, every particle continues to feel charges at larger and larger distances. The number of relevant interacting particles therefore grows with the size of the whole system.

2. Add screening.
Now suppose the charges are embedded in a neutral medium, such as a plasma or electrolyte, whose mobile opposite charges rearrange and screen the long-range electric field.
A simple effective screened interaction is the Yukawa form

$$\boxed{v_{\mathrm{scr}}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r}e^{-r/\lambda},}$$

where λ is the screening length.
For $r \ll \lambda$, the interaction is approximately Coulombic, while for $r \gg \lambda$, the exponential strongly suppresses it.
Assume $R \gg \lambda$ and consider a particle well inside the bulk. For a roughly uniform state, the number of particles in a spherical shell of radius r and thickness dr is

$$dN(r) = n\,4\pi r^2 dr.$$

The interaction energy associated with one particle is therefore estimated by

$$u_{\mathrm{one}} \simeq \int_0^\infty v_{\mathrm{scr}}(r)\,n\,4\pi r^2 dr.$$

When summing this over all particles, each pair would be counted twice. Thus

$$\frac{U_{\mathrm{scr}}}{N} \simeq \frac{1}{2}u_{\mathrm{one}}.$$

Hence

$$\begin{aligned}\frac{U_{\mathrm{scr}}}{N} &\simeq \frac{n}{2}\int_0^\infty \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r}e^{-r/\lambda}\,4\pi r^2 dr\\ &= 2\pi n\frac{1}{4\pi\varepsilon_0}q^2\int_0^\infty re^{-r/\lambda}\,dr.\end{aligned}$$

Using

$$\int_0^\infty re^{-r/\lambda}\,dr = \lambda^2,$$

we obtain

$$\boxed{\frac{U_{\mathrm{scr}}}{N} \simeq 2\pi n\frac{1}{4\pi\varepsilon_0}q^2\lambda^2}.$$

The important point is that this result contains no factor of the system radius R and no growing power of N. Therefore,

$$\boxed{U_{\mathrm{scr}} \propto N.}$$

Checks and physical lesson

The two cases differ by one physical idea:

$$\boxed{\begin{gathered}\text{unscreened: interaction range grows with the system}\\ \Longrightarrow U \sim N^{5/3},\\[4pt] \text{screened: interaction range remains finite, } r \sim \lambda\\ \Longrightarrow U \sim N.\end{gathered}}$$

Screening therefore converts the long-range Coulomb interaction into an effectively short-range interaction and restores a well-defined extensive thermodynamic scaling.
The numerical coefficient above is a simple uniform-density estimate; correlations between particles can change that coefficient, but not the basic extensivity argument when the screening length λ remains finite as N grows.

Checks and physical lesson

Screening makes each charge interact effectively only with a finite neighborhood of size λ, so adding more distant particles no longer increases the energy per particle. In a physical plasma or electrolyte the screening is supplied by other mobile charges or a neutralizing environment; a container or external pressure fixes the density. Under those conditions an ordinary thermodynamic equilibrium and an extensive energy become possible.

21. Two complementary descriptions of electrostatics

By the end of this week, electrostatics can be organized around two fields.

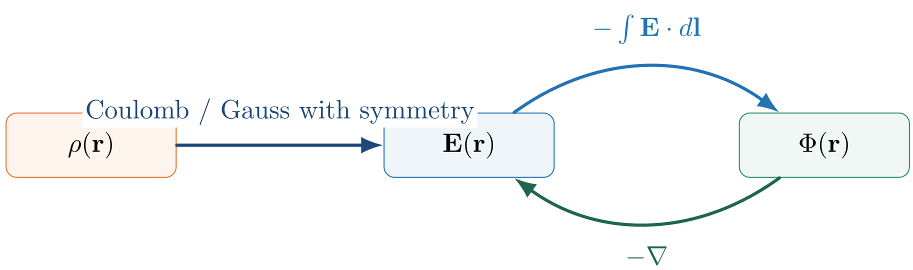
The vector description uses $\mathbf{E}$:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \times \mathbf{E} = 0}.$$

The scalar description uses Φ:

$$\boxed{\mathbf{E} = -\nabla\Phi}.$$

The first equation tells us where the sources of the field are. The second tells us that the electrostatic field is conservative. The potential then packages the field into one scalar function.



$$\text{source picture: } q, \lambda, \sigma, \rho(\mathbf{r})$$
$$\Downarrow$$
$$\text{field picture: } \nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \quad \nabla \times \mathbf{E} = 0$$
$$\Downarrow$$
$$\text{potential picture: } \mathbf{E} = -\nabla\Phi$$

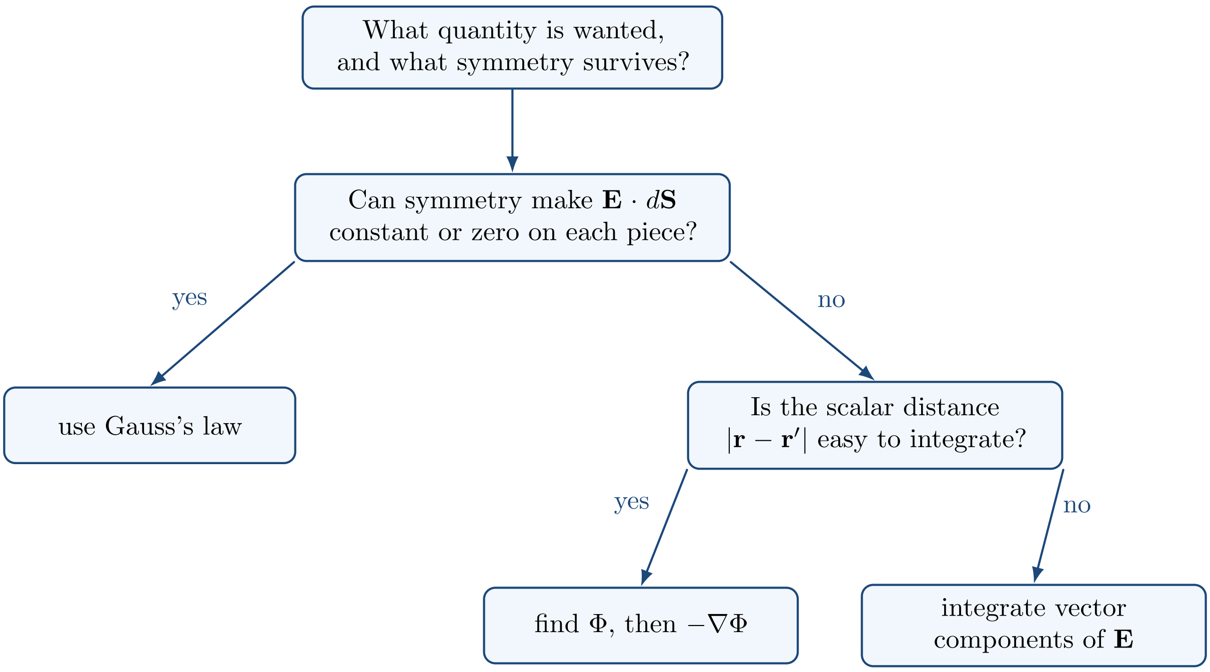
This is the first complete example in the course of the general Maxwell viewpoint: sources determine local field structure, integral laws connect that structure to boundaries, and potentials encode the curl-free part of the field efficiently.

22. Choosing the right method

A common difficulty is deciding whether to calculate the field directly, use Gauss's law, or calculate the potential first.

Method	**Best situation**	**Main idea**
Direct Coulomb integral	Arbitrary source with a manageable geometry	Add vector field contributions.
Gauss's law	Strong spherical, cylindrical, or planar symmetry	Use flux and enclosed charge.
Potential first	The scalar potential integral is simpler than the vector field integral	Calculate Φ, then use $\mathbf{E} = -\nabla\Phi$.

A good solution begins by identifying symmetry before writing a long integral.



Choose the method from the symmetry and the quantity you want, not from habit.

Solution strategy

Before doing algebra, write four short lines:

1. **Direction:** which components are allowed by symmetry?
2. **Dependence:** which coordinates can the magnitude depend on?
3. **Method:** Gauss, direct field, or potential first?
4. **Checks:** units, sign, continuity, near limit, and far limit.

These lines are not decoration. They often remove most of the calculation and make errors visible before the final answer.

23. Common points to keep clear

1. $\mathbf{r}'$ is a source position; $\mathbf{r}$ is an observation position.
2. The separation vector is $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and points from source to observation point.
3. Electric field is a vector. Potential is a scalar.
4. Superposition applies to both $\mathbf{E}$ and Φ, but vector addition is required for $\mathbf{E}$.
5. Gauss's law is always true, but it is easy to use for finding $\mathbf{E}$ only when symmetry is strong enough.
6. A Gaussian surface is imaginary. It does not need to coincide with a physical surface.
7. Q_{enc} includes only charge inside the Gaussian surface. Charges outside can still contribute to the local field, but their net contribution to the total flux through the closed surface is zero.
8. Zero enclosed charge does *not* imply $\mathbf{E} = 0$ everywhere on the Gaussian surface.
9. In electrostatics, $\nabla \times \mathbf{E} = 0$, so the electric field is conservative.
10. The zero of potential is a choice. Only potential differences determine physical work.
11. $\mathbf{E} = -\nabla\Phi$: the field points toward decreasing potential.
12. Do not confuse electric potential Φ (energy per unit charge) with electric potential energy $U = q\Phi$.
13. A region with $\mathbf{E} = 0$ has constant potential, but that constant need not be zero.
14. For localized sources, $\Phi(\infty) = 0$ is usually convenient. For infinite idealized sources, it may not be possible.
15. A charge outside a closed Gaussian surface can change $\mathbf{E}$ on the surface while contributing zero net flux through it.
16. Equal symmetry does not mean equal formulas: point, line, and plane sources produce different radial falloffs because the relevant boundary area grows differently.
17. A point where $\Phi = 0$ need not have $\mathbf{E} = 0$, and a point where $\mathbf{E} = 0$ need not have $\Phi = 0$.

24. Week 2 at a glance

The logical structure of this week is

$$\boxed{\begin{array}{c}\text{electric charge} \to \text{Coulomb force} \to \text{electric field} \to \text{superposition}\\ \Downarrow \\ \text{electric flux} \to \text{Gauss's law} \to \text{symmetry solutions} \\ \Downarrow \\ \text{conservative field} \to \text{electric potential} \to \mathbf{E} = -\nabla\Phi\end{array}}$$

The central relations are

$$\boxed{\mathbf{F} = q\mathbf{E}}$$

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,dV'}$$

$$\boxed{\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\varepsilon_0}}$$

$$\boxed{\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla\times\mathbf{E} = 0}$$

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV', \qquad \mathbf{E} = -\nabla\Phi}$$

25. Bridge to Week 3

We now have

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \mathbf{E} = -\nabla\Phi.$$

Substituting $\mathbf{E} = -\nabla\Phi$ into Gauss's law gives

$$\frac{\rho}{\varepsilon_0} = \nabla\cdot\mathbf{E} = -\nabla\cdot(\nabla\Phi) = -\nabla^2\Phi.$$

Therefore,

$$\boxed{\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}} \qquad \text{(Poisson's equation)}.$$

In a charge-free region, $\rho = 0$, so

$$\boxed{\nabla^2\Phi = 0} \qquad \text{(Laplace's equation)}.$$

In Week 3, the source description $\rho \mapsto \mathbf{E}$ will be reorganized as a boundary-value problem for the scalar potential. Poisson's and Laplace's equations, together with boundary conditions and uniqueness, will become the main tools for solving electrostatic geometries that are not accessible by symmetry alone.