

Electrodynamics & Relativity

Week 3: Potential Theory

Boundaries, conductors, uniqueness, and images

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Main idea. In Week 2 we usually knew the charges and calculated the field. Near a conductor, the surface charge is often unknown: it moves until the conductor reaches the required potential. So the central question changes to *given the charges in a region and the conditions on its boundary, what potential is possible inside?*

$$\text{sources} + \text{boundary data} \longrightarrow \Phi \longrightarrow \mathbf{E} = -\nabla\Phi.$$

Poisson's equation gives the local rule; **boundary conditions** select the physical solution; **uniqueness** tells us when our checked candidate is the only answer.

1 From Coulomb's law to a boundary-value problem

The potential Φ is potential energy per unit charge: a test charge q_0 has potential energy $U = q_0\Phi$ in a fixed external potential. A potential difference therefore measures the work per unit charge associated with moving between two points.

Week 2 established

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad \mathbf{E} = -\nabla\Phi.$$

The gradient collects the spatial slopes of the potential:

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}.$$

A partial derivative with respect to x means that we change x while holding the other coordinates fixed. The minus sign in $\mathbf{E} = -\nabla\Phi$ makes the electric field point toward decreasing potential.

Combining the two field relations gives

Key relations

$$\boxed{\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}} \quad (\text{Poisson's equation}), \quad \boxed{\nabla^2\Phi = 0} \quad \text{where } \rho = 0 \quad (\text{Laplace's equation}). \quad (1.1)$$

The symbol ∇^2 is called the **Laplacian**. In Cartesian coordinates it adds the second spatial derivatives:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}.$$

A function is called **harmonic** in a region if it has continuous **second derivatives** there and **satisfies** Laplace's equation throughout that region. Here "harmonic" describes a spatial property; it does not mean that the potential oscillates in time. For example, $\Phi = C - E_0x$ is harmonic and produces the uniform field $E_0\hat{\mathbf{x}}$.

1.1 What changed from Week 2?

For a known localized charge distribution in otherwise empty space,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad \Phi(\infty) = 0.$$

Here $\mathbf{r}$ is the observation point, where we want the potential; $\mathbf{r}'$ labels the source points over which we integrate.

Near an electrode (a conducting surface kept at a specified electric potential), a direct integral would require its unknown surface charge. A voltage condition such as $\Phi = V_0$ is much easier to state. We solve for Φ using that condition, then recover the surface charge from the field at the electrode.

Example — the same local charge density, different fields

Suppose the local volume charge density is the same, $\rho = \rho_0$. Can different boundary conditions still produce different fields?

Solution. Both

$$\Phi_1(x) = -\frac{\rho_0 x^2}{2\varepsilon_0}, \quad \Phi_2(x) = -\frac{\rho_0 x^2}{2\varepsilon_0} - E_0 x + C$$

satisfy the same Poisson equation. Their fields are

$$\mathbf{E}_1 = \frac{\rho_0 x}{\varepsilon_0} \hat{\mathbf{x}}, \quad \mathbf{E}_2 = \mathbf{E}_1 + E_0 \hat{\mathbf{x}}.$$

The constant C changes the reference potential. The term $-E_0 x$ changes the field and can represent the influence of remote electrodes. **Same local charge density does not mean the same field: the boundary data matter.**

More generally, if Φ_p is one solution of Poisson's equation, then

$$\Phi = \Phi_p + \Phi_h, \quad \nabla^2 \Phi_h = 0 \quad (1.2)$$

has the same volume charge density. The subscript “p” means a **particular solution**: one function that gives the required source. **The harmonic correction Φ_h carries the effect of the boundaries.**

This works because the equation is **linear**: differentiating a sum gives the sum of the derivatives. Consequently, potentials and fields from compatible source and boundary contributions can be added. This is **superposition**.

2 Begin with a piece of metal

A **conductor** contains charges that can move. In **electrostatic equilibrium**, the macroscopic charge distribution has stopped changing. We consider ideal conductors at rest, with vacuum around them and no sustained current.

2.1 Four facts to remember

Proof

1. The field inside the metal is zero.

If a nonzero electric field remained in the metal, it would push the mobile charges. They would continue to move, so the system would not be in equilibrium. Therefore

$$\mathbf{E}_{\text{metal}} = 0.$$

This refers to the *metal itself*. A cavity is not metal and may contain a nonzero field.

Proof

2. One connected conductor has one potential.

For two points A and B inside the same connected piece of metal,

$$\Phi(B) - \Phi(A) = - \int_A^B \mathbf{E} \cdot d\boldsymbol{\ell} = 0.$$

Hence

$$\Phi = V_c \quad \text{throughout one connected conductor.}$$

Its surface has the same value: the conductor is an **equipotential**.

Two disconnected conductors may have different potentials. If they are connected by an ideal wire, charge flows until their potentials become equal.

Proof

3. Excess charge lies on surfaces.

Take a small Gaussian surface completely inside the metal. Since $\mathbf{E} = 0$, its electric flux is zero. Gauss's law then gives zero enclosed *net* charge.

Therefore the bulk net charge density vanishes, and any excess charge must reside on a surface, including the surface of a cavity.

The metal still contains positive ions and electrons; their bulk charges simply balance.

Proof

4. The field just outside a smooth surface is perpendicular to it.

A tangential electric field would move charges along the surface. Therefore, in electrostatic equilibrium,

$$\mathbf{E}_{\parallel} = 0.$$

The remaining field is normal to the surface. Its magnitude is determined by the surface charge. We derive the precise relation in Section 7.

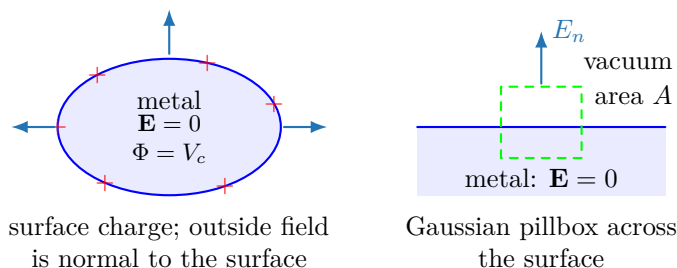


Figure 1: The four conductor facts in one picture. Inside the metal, $\mathbf{E} = 0$ and Φ is constant. Excess charge lies on the surface, and the field immediately outside is normal to it. The dashed pillbox is an imaginary Gaussian surface used later to relate E_n to the surface charge density.

Example — zero field does not mean zero potential

A metal object is held at 12 V. What are the potential and electric field at a point inside the metal? Does the answer depend on the shape?

Solution.

The entire connected conductor has the same potential,

$$\Phi = 12 \text{ V},$$

while the field inside is zero:

$$\Phi = 12 \text{ V}, \quad \mathbf{E} = 0.$$

This conclusion is independent of whether the conductor is spherical, flat, or irregular.

The electric field measures how the potential changes with position. A nonzero constant potential therefore gives zero field.

Physical meaning

Grounded means connected to a charge reservoir whose potential we choose as zero; charge may enter or leave.

Isolated means charge cannot enter or leave; its total charge is fixed, while its potential must be found.

Neutral means only that the *total* charge is zero.

These are different physical conditions.

3 What does the Laplacian mean physically?

If the potential depends only on x , Poisson's equation becomes

$$\Phi''(x) = -\rho(x)/\varepsilon_0.$$

Positive charge therefore makes the graph of $\Phi(x)$ bend downward.

In **several dimensions**, the equation constrains the **sum of the curvatures**, not each curvature separately. For example,

$$\Phi = A(x^2 - y^2)$$

has $\Phi_{xx} = 2A$ and $\Phi_{yy} = -2A$. The two curvatures cancel, so the function is curved but still harmonic. This is the basic geometry of a **saddle**.

3.1 Harmonic means “equal to the surrounding average”

For a harmonic potential, the cleanest statement is exact. If a whole ball of radius R around $\mathbf{r}_0$ is charge-free, then

$$\Phi(\mathbf{r}_0) = \frac{1}{4\pi R^2} \oint_{S_R} \Phi \, dS. \quad (3.1)$$

The value at the center equals the average value on every surrounding sphere that stays inside the charge-free region. **A harmonic potential cannot hide an isolated peak or well in the interior: its value is tied to the values around it.**

The same idea on a small grid

For a smooth function,

$$\Phi(x \pm h) = \Phi(x) \pm h\Phi'(x) + \frac{h^2}{2}\Phi''(x) + \dots$$

Adding opposite points cancels the first derivatives. In three dimensions, averaging over the six nearest neighbors gives

$$\frac{1}{6} \sum_{\text{six neighbors}} \Phi = \Phi(\mathbf{r}) + \frac{h^2}{6} \nabla^2 \Phi(\mathbf{r}) + O(h^4). \quad (3.2)$$

Thus, where $\nabla^2 \Phi = 0$, the central value is approximately the average of its nearest neighbors; the exact continuum version is Eq. (3.1). This same grid relation becomes the relaxation rule in Section ??.

Mean-value property on a sphere

Define the spherical average

$$M(R) = \frac{1}{4\pi} \int \Phi(\mathbf{r}_0 + R\hat{\mathbf{r}}) \, d\Omega.$$

Then

$$M'(R) = \frac{1}{4\pi} \int \nabla \Phi \cdot \hat{\mathbf{r}} \, d\Omega = \frac{1}{4\pi R^2} \oint_{S_R} \nabla \Phi \cdot \hat{\mathbf{n}} \, dS.$$

By the divergence theorem,

$$M'(R) = \frac{1}{4\pi R^2} \int_{B_R} \nabla^2 \Phi \, dV = 0,$$

because $\nabla^2 \Phi = 0$. Hence $M(R)$ is independent of R . Taking $R \rightarrow 0$ gives $M(R) = M(0) = \Phi(\mathbf{r}_0)$, so

$$\Phi(\mathbf{r}_0) = \frac{1}{4\pi R^2} \oint_{S_R} \Phi \, dS.$$

Physical meaning

Laplace's equation does not mean “no field.” It means that the directional curvatures balance. Charges outside the region can still create a strong harmonic field inside it.

4 Poisson's equation for a uniformly charged slab

Consider an infinite slab $-a < x < a$ with fixed uniform volume density $\rho_0 > 0$, no charge sheet at either face, and no applied external field. This models charges held in place, as in an ideal charged insulator.

Moving parallel to the slab does not change the problem, so $\Phi = \Phi(x)$. Reflection about its middle makes the potential even: $\Phi(-x) = \Phi(x)$, hence $\Phi'(0) = 0$. Choose the reference $\Phi(0) = 0$.

Inside. Integrating $\Phi'' = -\rho_0/\varepsilon_0$ twice gives

$$\Phi = -\frac{\rho_0 x^2}{2\varepsilon_0}, \quad E_x = \frac{\rho_0 x}{\varepsilon_0}.$$

Outside. There is no volume charge, so $\Phi'' = 0$ and the potential is linear on each side. **No surface charge sheet $\Rightarrow$ the potential and its normal slope are continuous.** Section 8 derives these matching rules.

At $x = a$ the exterior value and slope must therefore be $-\rho_0 a^2/(2\varepsilon_0)$ and $-\rho_0 a/\varepsilon_0$. Reflection gives the left side:

$$\Phi(x) = \begin{cases} -\frac{\rho_0 x^2}{2\varepsilon_0}, & |x| \leq a, \\ -\frac{\rho_0 a}{\varepsilon_0} \left(|x| - \frac{a}{2} \right), & |x| \geq a. \end{cases} \quad (4.1)$$

Thus

$$E_x(x) = \begin{cases} -\rho_0 a/\varepsilon_0, & x < -a, \\ \rho_0 x/\varepsilon_0, & |x| < a, \\ +\rho_0 a/\varepsilon_0, & x > a. \end{cases}$$

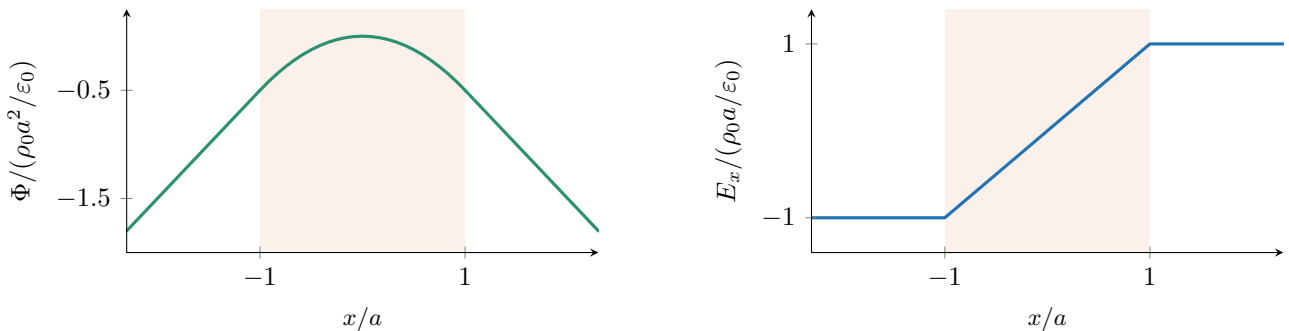


Figure 2: Shading marks the charged slab. Potential and slope match at its faces; curvature changes.

Check / physical lesson

Check the potential and slope at $x = \pm a$. Independently, a Gaussian pillbox with end area A outside both faces encloses charge $2aA\rho_0$. Gauss's law gives $2EA = 2aA\rho_0/\varepsilon_0$, agreeing with the exterior field. This field does not decay at infinity. Therefore the potential cannot approach a finite constant there: imposing $\Phi(\infty) = 0$ would be inconsistent with this infinite-slab model.

5 Boundary data: what must be specified?

First draw the region being solved. Its boundary may contain electrode surfaces, interfaces, symmetry surfaces introduced to simplify the calculation, and a condition at infinity if the region is unbounded.

A **normal** is perpendicular to a surface. The outward unit normal $\hat{n}_\Omega$ has length one and points *out of the solution region*. The **normal derivative** is the potential slope in that direction:

$$\frac{\partial \Phi}{\partial n_\Omega} = \hat{n}_\Omega \cdot \nabla \Phi, \quad E_{n_\Omega} = -\frac{\partial \Phi}{\partial n_\Omega}. \quad (5.1)$$

Type	Mathematical data	Meaning
Dirichlet	$\Phi = f$	Potential prescribed at each boundary point.
Neumann	$\partial_{n\Omega} \Phi = g$	Normal slope prescribed; outward electric field is $-g$.
Mixed	Dirichlet on one part, Neumann on another	Each boundary piece has its own condition.
Robin	$\alpha\Phi + \beta\partial_{n\Omega} \Phi = h$	A relation between value and normal slope on the same boundary piece.

The given functions f, g, h may vary along the boundary. The coefficients in a Robin condition must have compatible units. We mainly use Dirichlet and Neumann conditions this week.

Translate physical words into equations

Physical statement	Equation to write
Grounded conductor	$\Phi = 0$ on its surface, taking ground as the reference.
Conductor maintained at voltage V_c	$\Phi = V_c$ on its entire connected surface.
Isolated conductor with total charge Q_c	$\Phi = C_c$, with C_c unknown, and $\int \sigma dS = Q_c$.
Mirror-symmetry plane where potential is even	$\partial_n \Phi = 0$.
Known normal field E_n	$\partial_n \Phi = -E_n$, with the normal explicitly stated.
No point or line source at the origin	Keep the solution regular there; discard unphysical divergent terms.
Localized sources and no applied far field	Usually $\Phi \rightarrow 0$ as $r \rightarrow \infty$.
Prescribed uniform far field $\mathbf{E}_0$	Specify the approach of $\Phi + \mathbf{E}_0 \cdot \mathbf{r}$ to its reference constant.

Regular means finite and sufficiently smooth where no physical source requires a divergence. For example, a $1/r$ term diverges at $r = 0$ and represents a point source there; it cannot be included at a charge-free origin.

An insulator does not automatically impose $\partial_n \Phi = 0$. Electric fields can pass through insulating material.

5.1 A restriction on Neumann boundary conditions

Suppose we prescribe the normal derivative of the potential on the whole boundary,

$$g = \frac{\partial \Phi}{\partial n}.$$

Can we choose g arbitrarily? **No.** Poisson's equation itself imposes one condition.

Integrating

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$$

over the volume and using the divergence theorem gives

$$\oint_{\partial\Omega} \frac{\partial \Phi}{\partial n} dS = -\frac{Q_\Omega}{\epsilon_0}.$$

Physical meaning

This is simply Gauss's law written in terms of the potential. The total normal derivative on the boundary must agree with the total charge enclosed by the boundary.

Therefore, in a charge-free region,

$$Q_\Omega = 0 \quad \implies \quad \oint_{\partial\Omega} \frac{\partial \Phi}{\partial n} dS = 0.$$

The outward derivatives on different parts of the boundary must therefore balance. We cannot prescribe arbitrary values independently.

Example — a simple one-dimensional check

Consider an empty interval $0 < x < d$. Since $\rho = 0$,

$$\Phi''(x) = 0,$$

so

$$\Phi(x) = A + Bx, \quad \frac{d\Phi}{dx} = B.$$

Solution.

For Neumann boundary conditions we need the derivative in the *outward normal direction*,

$$\frac{\partial \Phi}{\partial n} = \nabla \Phi \cdot \hat{\mathbf{n}}.$$

At the left end, $x = 0$, the outward normal is $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$, so

$$\left. \frac{\partial \Phi}{\partial n} \right|_{x=0} = (B\hat{\mathbf{x}}) \cdot (-\hat{\mathbf{x}}) = -B.$$

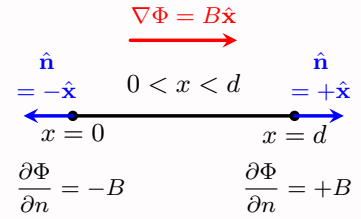
At the right end, $x = d$, the outward normal is $\hat{\mathbf{n}} = +\hat{\mathbf{x}}$, so

$$\left. \frac{\partial \Phi}{\partial n} \right|_{x=d} = (B\hat{\mathbf{x}}) \cdot (+\hat{\mathbf{x}}) = +B.$$

Thus

$$\left. \frac{\partial \Phi}{\partial n} \right|_{x=0} + \left. \frac{\partial \Phi}{\partial n} \right|_{x=d} = 0,$$

as required for a region containing no charge.

**Derivation**

One more property of pure Neumann conditions Neumann conditions specify derivatives of Φ , not its absolute value. Therefore, if Φ is a solution, then

$$\Phi + C$$

is also a solution for any constant C .

Thus a pure Neumann problem determines the potential only **up to an additive constant**. We may fix that constant by choosing, for example, $\Phi = 0$ at one reference point.

6 Parallel plates: imposing conditions step by step**6.1 An empty gap with fixed voltages**

Let $0 < x < d$ be the vacuum gap between infinite plates held at $\Phi(0) = V_0$ and $\Phi(d) = 0$. Ignore edge effects. The full problem is unchanged by translations parallel to the plates, so $\Phi = \Phi(x)$.

Laplace's equation gives $\Phi'' = 0$, with general solution $Ax + B$. Apply each boundary condition separately:

$$\Phi(0) = B = V_0, \quad \Phi(d) = Ad + B = 0 \implies A = -V_0/d.$$

Hence

$$\Phi(x) = V_0 \left(1 - \frac{x}{d}\right), \quad \mathbf{E} = \frac{V_0}{d} \hat{\mathbf{x}}. \quad (6.1)$$

For $V_0 > 0$, the field points from higher to lower potential.

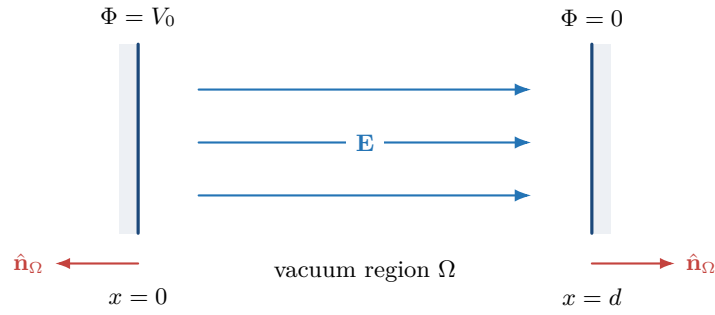


Figure 3: The outward normals of the vacuum gap point into the metal. They are opposite to the normals from each metal face into the gap.

6.2 The same plates with charge between them

Example — a space charge can reverse the field

Keep the plate voltages, but add prescribed immobile uniform volume charge $\rho_0 > 0$ in the gap. The charges must be held in place; freely moving charge would generally rearrange. Find the potential and ask when the field changes direction inside the gap.

Solution. Integrating Poisson's equation gives

$$\Phi = -\frac{\rho_0 x^2}{2\varepsilon_0} + Ax + B.$$

The left plate fixes $B = V_0$. The right plate fixes $A = -V_0/d + \rho_0 d/(2\varepsilon_0)$. Therefore

$$\Phi(x) = V_0 \left(1 - \frac{x}{d}\right) + \frac{\rho_0}{2\varepsilon_0} x(d-x), \quad (6.2)$$

$$E_x(x) = \frac{V_0}{d} + \frac{\rho_0}{\varepsilon_0} \left(x - \frac{d}{2}\right).$$

The extra term changes the interior field while leaving both boundary voltages unchanged.

For $V_0 > 0$, the field vanishes at

$$x_* = \frac{d}{2} - \frac{\varepsilon_0 V_0}{\rho_0 d}.$$

This point is inside the gap exactly when $\rho_0 d^2 > 2\varepsilon_0 V_0$. The potential has a maximum there since $\Phi'' < 0$. There is *no conflict* with the maximum principle introduced next: **this region contains charge and is not harmonic.**

Check. Setting $\rho_0 = 0$ recovers the empty-gap solution. The quantity $\rho_0 d^2/\varepsilon_0$ has units of voltage. Also, $E_x(d) - E_x(0) = \rho_0 d/\varepsilon_0$, which is Gauss's law per unit plate area.

6.3 Mixed boundary conditions: one voltage and one field

We now solve a problem with two different types of boundary conditions. Suppose the potential is fixed at the left boundary, $\Phi(0) = 0$, while the electric field is fixed at the right boundary, $E_x(d) = E_d$. The first is a **Dirichlet condition**: it specifies the value of the potential.

The second is a **Neumann condition**: it specifies a derivative of the potential. Since

$$E_x = -\frac{d\Phi}{dx},$$

the condition $E_x(d) = E_d$ means

$$\Phi'(d) = -E_d.$$

Physical meaning

How to use a Neumann condition. A given electric field at a boundary must first be translated into a derivative of the potential using $\mathbf{E} = -\nabla\Phi$.

At the right boundary $x = d$, the outward normal is $+\hat{\mathbf{x}}$, so here $\partial\Phi/\partial n = \Phi'(d)$.

For a uniform charge density ρ_0 , Integrating Poisson's equation gives:

$$\Phi(x) = -\frac{\rho_0}{2\varepsilon_0}x^2 + Ax + B.$$

We now determine the two constants A and B from the two boundary conditions.

Example — applying one Dirichlet and one Neumann condition

Solution.

1. **Apply the voltage condition at $x = 0$:** $\Phi(0) = 0$. Therefore $B = 0$.
2. **Apply the field condition at $x = d$:** First differentiate the potential:

$$\Phi'(x) = -\frac{\rho_0}{\varepsilon_0}x + A.$$

Because $E_x = -\Phi'$,

$$E_x(d) = -\Phi'(d) = \frac{\rho_0 d}{\varepsilon_0} - A.$$

Setting this equal to the prescribed field E_d gives

$$E_d = \frac{\rho_0 d}{\varepsilon_0} - A,$$

so

$$A = \frac{\rho_0 d}{\varepsilon_0} - E_d.$$

Hence

$$\Phi(x) = -\frac{\rho_0 x^2}{2\varepsilon_0} + \left(\frac{\rho_0 d}{\varepsilon_0} - E_d\right)x.$$

The potential at the right boundary was *not* prescribed. It follows from the solution:

$$\Phi(d) = \frac{\rho_0 d^2}{2\varepsilon_0} - E_d d.$$

Physical meaning

What did we learn? A second-order equation needs two boundary conditions. Here we used

$$\Phi(0) = 0 \quad \text{and} \quad E_x(d) = E_d.$$

These already determine the complete solution. Therefore $\Phi(d)$ is an **output**; it cannot also be chosen independently.

7 Boundary conditions from a pillbox and a small loop

Let a surface separate vacuum regions 1 and 2. Choose a unit normal $\hat{\mathbf{n}}$ from region 1 to region 2. An ordinary surface charge density σ may reside on the surface.

A **Gaussian pillbox** is an imaginary thin closed box that crosses the surface. Its two large faces have area A . Electric flux is the surface integral $\int \mathbf{E} \cdot \hat{\mathbf{n}} dS$: it counts the signed normal field over the surface.

Gauss's law gives

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{n}} A = \frac{\sigma A}{\varepsilon_0}, \quad \boxed{E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0}}. \quad (7.1)$$

The side area tends to zero as the pillbox becomes thinner, so its flux vanishes when the one-sided fields remain finite.

A narrow rectangular loop crossing the interface gives

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \implies \boxed{\mathbf{E}_{2\parallel} = \mathbf{E}_{1\parallel}}. \quad (7.2)$$

The short sides contribute vanishingly little; the long sides have opposite orientations. **An ordinary surface charge changes the normal field, not the tangential field.**

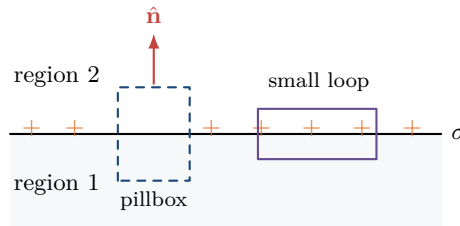


Figure 4: The pillbox determines the jump in the normal field. The loop determines continuity of the tangential field. Both calculations use a stated orientation.

At a metal–vacuum interface, choose $\hat{\mathbf{n}}_c$ to point *out of the metal into the vacuum*. Since the metal field is zero,

$$\boxed{\mathbf{E}_{\text{vac}} = \frac{\sigma}{\varepsilon_0} \hat{\mathbf{n}}_c, \quad \sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial n_c}}. \quad (7.3)$$

A negative surface charge produces a field directed into the metal.

Example — read the surface charge from the field

The field just outside a metal surface is 1000 V/m and points into the metal. Find the surface charge density.

Solution. Choose the unit normal to point out of the metal. Then $E_n = -1000$ V/m, so

$$\boxed{\sigma = \varepsilon_0 E_n = (8.85 \times 10^{-12})(-1000) = -8.85 \times 10^{-9} \text{ C/m}^2}.$$

Fix the normal direction first; the sign of E_n then fixes the sign of σ .

Method / strategy

At a metal boundary of a vacuum solution region, $\hat{\mathbf{n}}_\Omega = -\hat{\mathbf{n}}_c$. Consequently,

$$\sigma = +\varepsilon_0 \frac{\partial \Phi}{\partial n_\Omega}.$$

Draw both normal arrows before using a surface-charge formula. The physical result is the same; the derivative sign follows the chosen direction.

Example — surface charge on the two faces of the plates

Two large conducting plates are located at $x = 0$ and $x = d$. The gap between them contains a uniform volume charge density ρ_0 . The electric field in the gap was found in Section 6:

$$E_x(x) = \frac{V_0}{d} + \frac{\rho_0}{\varepsilon_0} \left(x - \frac{d}{2} \right).$$

Problem. Find the surface charge densities σ_L and σ_R on the two metal faces that look directly into the charged gap.

At a conductor surface,

$$\sigma = \varepsilon_0 \mathbf{E}_{\text{out}} \cdot \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ points *out of the metal*. Thus we need the field just outside each conductor and the correct normal direction.

Solution.

Left plate, $x = 0$.

The normal from the left conductor into the gap points in the positive x direction:

$$\hat{\mathbf{n}}_L = +\hat{\mathbf{x}}.$$

The field immediately outside this conductor is

$$E_x(0) = \frac{V_0}{d} - \frac{\rho_0 d}{2\varepsilon_0}.$$

Therefore

$$\sigma_L = \varepsilon_0 \mathbf{E}(0) \cdot \hat{\mathbf{n}}_L = \varepsilon_0 E_x(0),$$

and hence

$$\sigma_L = \frac{\varepsilon_0 V_0}{d} - \frac{\rho_0 d}{2}.$$

Right plate, $x = d$.

Here the normal from the conductor into the gap points in the negative x direction:

$$\hat{\mathbf{n}}_R = -\hat{\mathbf{x}}.$$

The field immediately outside this conductor is

$$E_x(d) = \frac{V_0}{d} + \frac{\rho_0 d}{2\varepsilon_0}.$$

Therefore

$$\sigma_R = \varepsilon_0 \mathbf{E}(d) \cdot \hat{\mathbf{n}}_R = -\varepsilon_0 E_x(d),$$

so

$$\sigma_R = -\frac{\varepsilon_0 V_0}{d} - \frac{\rho_0 d}{2}.$$

Physical meaning

Check. Adding the two facing surface charges gives

$$\sigma_L + \sigma_R = -\rho_0 d.$$

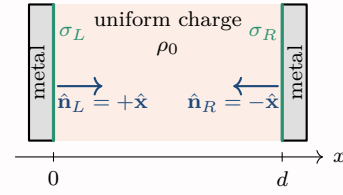
But the volume charge in the gap per unit plate area is

$$\frac{Q_{\text{gap}}}{A} = \rho_0 d.$$

Therefore

$$\sigma_L + \sigma_R + \rho_0 d = 0.$$

So the total charge on the two facing metal surfaces exactly balances the volume charge in the gap. The two terms proportional to V_0 cancel each other.



These are only the charges on the *facing* surfaces. The charges on the outer faces depend on the electric field outside the apparatus and cannot be determined from the solution inside the gap.

8 Matching potentials across an interface

To **match** two solutions means to connect them using the physical conditions at their common interface. Across a short path,

$$\Phi_2 - \Phi_1 = - \int_1^2 \mathbf{E} \cdot d\mathbf{\ell}.$$

If the one-sided field remains finite, the integral tends to zero as the path length tends to zero.

Ordinary charge sheet: Φ stays continuous, but its normal derivative jumps.

$$\boxed{\Phi_2 = \Phi_1, \quad \frac{\partial \Phi_2}{\partial n} - \frac{\partial \Phi_1}{\partial n} = -\frac{\sigma}{\epsilon_0}.} \quad (8.1)$$

Both derivatives are taken along the *same* normal.

Example — a charged sheet between grounded plates

Let $\Phi(0) = \Phi(d) = 0$ and place a fixed sheet of charge σ at $x = s$, where $0 < s < d$. There is no volume charge away from the sheet. Find the potential, field, and induced charge on the two plates.

Solution. Solve Laplace's equation on the two separate intervals:

$$\Phi_{<}(x) = Ax, \quad \Phi_{>}(x) = B(d - x).$$

These forms already satisfy the two grounded boundaries. Continuity and the derivative jump at s give

$$As = B(d - s), \quad -B - A = -\frac{\sigma}{\epsilon_0}.$$

The second equation follows because $\Phi'_{>} = -B$ and $\Phi'_{<} = A$. Solving,

$$\boxed{\Phi(x) = \frac{\sigma}{\epsilon_0 d} \begin{cases} (d - s)x, & 0 \leq x \leq s, \\ s(d - x), & s \leq x \leq d. \end{cases}} \quad (8.2)$$

The fields are

$$E_{<} = -\frac{\sigma(d - s)}{\epsilon_0 d}, \quad E_{>} = +\frac{\sigma s}{\epsilon_0 d}.$$

For $\sigma > 0$, both point away from the sheet.

The facing metal charges are

$$\sigma_L = -\sigma \frac{d - s}{d}, \quad \sigma_R = -\sigma \frac{s}{d}.$$

They sum to $-\sigma$. A centered sheet shares its compensating charge equally. As the sheet approaches the left plate, almost all the compensating charge appears there.

The sharp change of slope at the sheet is the potential signature of its surface charge.

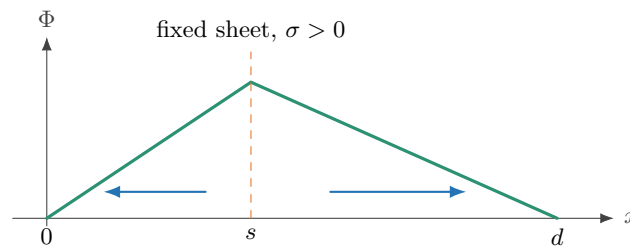


Figure 5: A positive sheet creates a tent-shaped potential. The potential is continuous, but its slope jumps. The maximum occurs at a charge sheet, outside the assumptions of the harmonic maximum principle.

9 Cavities, shielding, and charge bookkeeping

A connected piece of metal can have an outer surface and several inner cavity surfaces. All have the same potential V_c . The way the metal is connected and the cavities are enclosed is part of the physical boundary information.

Empty closed cavity. If a cavity contains no charge, its potential is harmonic and equals V_c everywhere on its wall. The maximum principle forces $\Phi = V_c$ throughout the cavity: the electric field inside is zero.

External static fields may change V_c and the exterior surface charge without producing a field in the empty cavity. The cavity must be fully enclosed by the conductor.

Charge in a cavity. If a charge q lies inside, the cavity field is generally nonzero. Choose a Gaussian surface entirely in the metal and surrounding the cavity. Its field and flux are zero, so

$$\boxed{Q_{\text{inner}} = -q.} \quad (9.1)$$

This determines the total inner charge without determining how that charge is distributed.

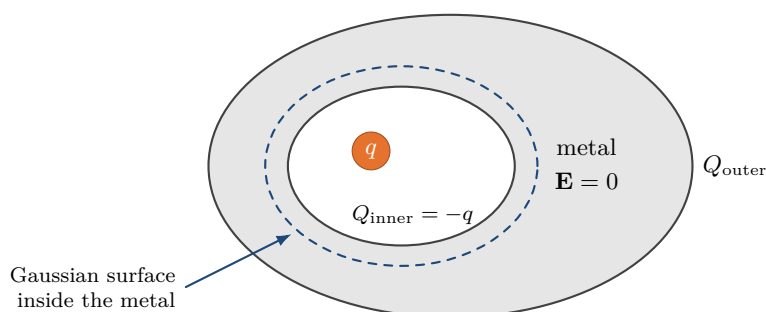


Figure 6: The Gaussian surface lies in the metal, so its flux is zero. An irregular cavity and an off-center charge change the distribution on the inner wall, but not its total charge $-q$.

For an isolated conductor with total charge Q_c ,

$$Q_{\text{inner}} + Q_{\text{outer}} = Q_c \implies \boxed{Q_{\text{outer}} = Q_c + q.}$$

For a grounded conductor with no external charges or applied field, and $\Phi \rightarrow 0$ at infinity, the exterior solution is zero by uniqueness. Then $Q_{\text{outer}} = 0$, and ground supplies the inner charge $-q$.

Physical meaning

An empty cavity cut from a uniformly charged *insulator*, as in Week 2, can contain a nonzero field. Those remaining charges are fixed, and the cavity wall is not required to have constant potential. Electrostatic shielding relies on a conducting enclosure.

10 Uniqueness: why checking a candidate is enough

Uniqueness means that the same complete physical data cannot give two different solutions. It is separate from **existence**, which asks whether any solution satisfying the data exists.

Uniqueness turns verification into a solution method: if one candidate has the correct equation, sources, and boundary data, there is no second physical answer.

10.1 Green's first identity

Derivation of Green's first identity

The product rule gives

$$\nabla \cdot (U \nabla U) = |\nabla U|^2 + U \nabla^2 U.$$

Integrating over the region and applying the divergence theorem gives

$$\int_{\Omega} |\nabla U|^2 dV = \oint_{\partial\Omega} U \frac{\partial U}{\partial n_{\Omega}} dS - \int_{\Omega} U \nabla^2 U dV. \quad (10.1)$$

This is **Green's first identity**: a useful rearrangement of the product rule and the divergence theorem, not a new physical law. We assume sufficient smoothness and a connected region with piecewise smooth boundary.

10.2 Dirichlet, Neumann, and mixed uniqueness

Proof

Suppose Φ_1 and Φ_2 solve the same Poisson equation with the same sources. Their difference $U = \Phi_1 - \Phi_2$ is harmonic: $\nabla^2 U = 0$.

If both have the same Dirichlet data, then $U = 0$ on the boundary. Green's identity reduces to

$$\int_{\Omega} |\nabla U|^2 dV = 0.$$

A nonnegative quantity cannot integrate to zero through cancellation between positive and negative contributions. Thus $\nabla U = 0$, so U is constant. Its zero boundary value makes that constant zero.

Key relations

With a fixed charge density and specified potential on the whole boundary of a bounded region, the potential is unique.

If point charges are present, both candidates must have the same Coulomb singularities at the same positions. Those singularities cancel in their difference. An extra source cannot be hidden inside a candidate solution. For matching Neumann data, $\partial_{n_{\Omega}} U = 0$ on the boundary. The same identity proves $\nabla U = 0$, but leaves an additive constant. Compatible Neumann data therefore determine a unique field; a reference potential fixes the remaining constant.

For mixed data, $U = 0$ on the Dirichlet part and $\partial_{n_{\Omega}} U = 0$ on the Neumann part. The boundary integral again vanishes. A Dirichlet portion of nonzero extent fixes the constant.

10.3 The boundary at infinity matters

For an exterior problem, apply the identity inside a large sphere and then let its radius grow. If the difference of two candidates satisfies $U = O(1/r)$ and $\nabla U = O(1/r^2)$, its far-surface contribution vanishes as $O(1/r)$. Other far-field conditions require the corresponding control.

Example — a grounded plane alone is insufficient

Show why the condition $\Phi = 0$ on the plane $z = 0$ does not by itself select a unique solution in the upper half-space.

Solution. In $z > 0$, both $\Phi = 0$ and $\Phi = -E_0 z$ are harmonic and vanish at $z = 0$. They have different fields far away.

Requiring $\Phi \rightarrow 0$ selects the first. Requiring a uniform applied field $\mathbf{E} \rightarrow E_0 \hat{\mathbf{z}}$, with the corresponding potential behavior, selects the second. Uniqueness requires the complete boundary data, including infinity.

11 Spherical boundaries: solve the radial equation first

Consider two concentric conducting spheres of radii $a < b$, held at fixed potentials V_a (inner) and V_b (outer). The region between them is vacuum.

Because the conductors are concentric, the geometry looks the same in every direction from the center.

Therefore the potential can depend only on the distance r :

$$\Phi = \Phi(r).$$

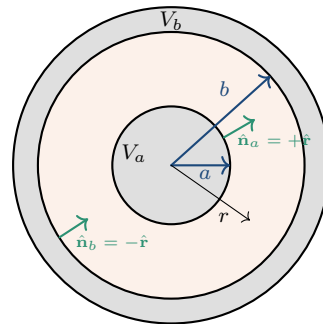
The boundary conditions are

$$\Phi(a) = V_a, \quad \Phi(b) = V_b.$$

Our goal is to find the potential and electric field in

$$a < r < b,$$

and then determine the charge on the two facing conductor surfaces.



Method / strategy

Method for a spherically symmetric boundary problem

$$\text{symmetry} \longrightarrow \Phi(r) \longrightarrow \text{radial Laplace equation} \longrightarrow \text{boundary conditions}$$

Once $\Phi(r)$ is known, use

$$\mathbf{E} = -\nabla\Phi$$

and then

$$\sigma = \varepsilon_0 \mathbf{E}_{\text{out}} \cdot \hat{\mathbf{n}}$$

to find the surface charge.

11.1 Solve Laplace's equation

There is no charge in the region $a < r < b$, so

$$\nabla^2\Phi = 0.$$

For a function that depends only on r , the Laplacian becomes

$$\nabla^2\Phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right).$$

Therefore

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 0.$$

Multiply by r^2 and Integrating once,

$$r^2 \frac{d\Phi}{dr} = C,$$

so

$$\frac{d\Phi}{dr} = \frac{C}{r^2}.$$

Integrating again,

$$\Phi(r) = A + \frac{B}{r}.$$

11.2 Apply the two boundary voltages

The boundary conditions are $\Phi(a) = V_a, \Phi(b) = V_b$. Using $\Phi(r) = A + \frac{B}{r}$, we obtain

$$A + \frac{B}{a} = V_a, \quad A + \frac{B}{b} = V_b.$$

Subtracting the two equations gives

$$B \left(\frac{1}{a} - \frac{1}{b} \right) = V_a - V_b.$$

Since $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$, we find $B = \frac{ab(V_a - V_b)}{b-a}$, and then $A = \frac{bV_b - aV_a}{b-a}$.

Therefore

$$\Phi(r) = \frac{bV_b - aV_a}{b-a} + \frac{ab(V_a - V_b)}{(b-a)r}, \quad a < r < b. \quad (11.1)$$

The same result can also be written as

$$\Phi(r) = \frac{a(b-r)}{r(b-a)} V_a + \frac{b(r-a)}{r(b-a)} V_b.$$

The two coefficients are nonnegative for $a < r < b$ and add to one. Therefore the potential everywhere between the spheres lies between the two boundary voltages.

11.3 Electric field and surface charge

$$\mathbf{E}(r) = -\nabla\Phi = \frac{B}{r^2} \hat{\mathbf{r}}.$$

The field is therefore radial and falls as $1/r^2$.

We now use

$$\sigma = \varepsilon_0 \mathbf{E}_{\text{out}} \cdot \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ points from the conductor into the vacuum region.

Inner sphere, $r = a$.

The normal from the inner conductor into the gap points outward: $\hat{\mathbf{n}}_a = +\hat{\mathbf{r}}$. Therefore

$$\sigma_a = \varepsilon_0 \mathbf{E}(a) \cdot \hat{\mathbf{r}} = \frac{\varepsilon_0 B}{a^2}.$$

Thus

$$\sigma_a = \frac{\varepsilon_0 B}{a^2}.$$

Inner face of the outer conductor, $r = b$.

Here the conductor lies outside the gap. The normal from this conductor into the gap therefore points toward smaller r : $\hat{\mathbf{n}}_b = -\hat{\mathbf{r}}$.

Hence

$$\sigma_{b,\text{inner}} = \varepsilon_0 \mathbf{E}(b) \cdot (-\hat{\mathbf{r}}) = -\frac{\varepsilon_0 B}{b^2}.$$

Therefore

$$\sigma_{b,\text{inner}} = -\frac{\varepsilon_0 B}{b^2}.$$

Physical meaning**Charge check.**

The total charge on the inner sphere is

$$Q_a = 4\pi a^2 \sigma_a = 4\pi \epsilon_0 B.$$

The total charge on the facing surface of the outer conductor is

$$Q_{b,\text{inner}} = 4\pi b^2 \sigma_{b,\text{inner}} = -4\pi \epsilon_0 B.$$

Thus

$$Q_a + Q_{b,\text{inner}} = 0.$$

The two facing surfaces carry equal and opposite total charges, although their *surface charge densities* are different because their areas are different.

Check / physical lesson**Useful checks**

- If $V_a = V_b$, then $B = 0$, so

$$\mathbf{E} = 0$$

and the potential is constant.

- If the outer boundary is grounded, $V_b = 0$, and then moved very far away, $b \rightarrow \infty$, we obtain

$$\Phi(r) \rightarrow V_a \frac{a}{r}.$$

This is the familiar potential outside an isolated conducting sphere.

- For a narrow gap

$$d = b - a \ll a,$$

write

$$r = a + x, \quad 0 < x < d.$$

Then, to leading order,

$$\Phi(r) \simeq V_a + \frac{V_b - V_a}{d} x.$$

Locally, the curved surfaces look flat and the result reduces to the parallel-plate potential.

Example — which radial terms are physically allowed?

For spherical symmetry,

$$\Phi(r) = A + \frac{B}{r}.$$

Problem. Which terms are allowed in two different geometries: (i) inside a charge-free sphere containing the origin, and (ii) outside a single isolated conducting sphere?

Solution.**1. Inside a charge-free sphere, $0 \leq r < a$.**

Let

$$\Phi(a) = V_a.$$

Because B/r diverges at $r = 0$, regularity requires

$$B = 0.$$

Hence

$$\Phi(r) = V_a.$$

A nonzero B/r term would correspond to a point charge at the origin.

2. Separate case: outside one isolated sphere, $r > a$.

Now there is no outer conductor. Take

$$\Phi(a) = V_a, \quad \Phi(\infty) = 0.$$

The origin is outside the solution region, so B/r is allowed. Instead, the condition at infinity requires

$$A = 0.$$

Then

$$\frac{B}{a} = V_a \quad \Rightarrow \quad B = V_a a,$$

and therefore

$$\Phi(r) = V_a \frac{a}{r}.$$

Physical meaning

A term is allowed or excluded according to the **solution region and its boundary conditions**. A divergence matters only if it occurs inside the region being solved.

12 Cylindrical boundaries: solve the radial equation first

Consider two long concentric conducting cylinders of radii $a < b$, held at fixed potentials V_a (inner) and V_b (outer). The region between them is vacuum.

We use cylindrical coordinates (ρ, ϕ, z) , where ρ is the distance from the common axis, ϕ is the azimuthal angle, and z runs along the cylinders.

Because the cylinders are very long and concentric, the system is unchanged by rotation in ϕ or translation in z .

Therefore the potential depends only on the radial coordinate:

$$\Phi = \Phi(\rho).$$

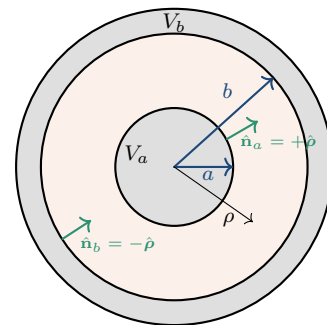
The boundary conditions are

$$\Phi(a) = V_a, \quad \Phi(b) = V_b.$$

Our goal is to find $\Phi(\rho)$ and $\mathbf{E}(\rho)$ in

$$a < \rho < b,$$

and then determine the charges on the two facing surfaces.

**Method / strategy****Method for a cylindrically symmetric boundary problem**

$$\text{symmetry} \rightarrow \Phi(\rho) \rightarrow \text{radial Laplace equation} \rightarrow \text{boundary conditions}$$

Once $\Phi(\rho)$ is known, use

$$\mathbf{E} = -\nabla\Phi, \quad \sigma = \varepsilon_0 \mathbf{E}_{\text{out}} \cdot \hat{\mathbf{n}}.$$

12.1 Solve Laplace's equation

There is no charge in the region $a < \rho < b$, so

$$\nabla^2 \Phi = 0.$$

In cylindrical coordinates,

$$\nabla^2 \Phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}.$$

Here Φ is independent of ϕ and z , so

$$\nabla^2 \Phi = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{d\Phi}{d\rho} \right) = 0.$$

Multiplying by ρ and integrating once,

$$\rho \frac{d\Phi}{d\rho} = C,$$

so

$$\frac{d\Phi}{d\rho} = \frac{C}{\rho}.$$

Integrating again,

$$\Phi(\rho) = A + B \ln \rho.$$

12.2 Apply the two boundary voltages

Using

$$\Phi(a) = V_a, \quad \Phi(b) = V_b,$$

we obtain

$$A + B \ln a = V_a, \quad A + B \ln b = V_b.$$

Subtracting,

$$B \ln \left(\frac{b}{a} \right) = V_b - V_a,$$

so

$$B = \frac{V_b - V_a}{\ln(b/a)}.$$

A compact form of the solution is therefore

$$\Phi(\rho) = V_a + (V_b - V_a) \frac{\ln(\rho/a)}{\ln(b/a)}, \quad a < \rho < b. \quad (12.1)$$

At $\rho = a$ the logarithm vanishes, while at $\rho = b$ it equals $\ln(b/a)$, so the two boundary values are recovered immediately.

12.3 Electric field and surface charge

Differentiating,

$$\frac{d\Phi}{d\rho} = \frac{V_b - V_a}{\rho \ln(b/a)}.$$

Hence

$$\mathbf{E}(\rho) = \frac{V_a - V_b}{\rho \ln(b/a)} \hat{\rho}.$$

Thus the cylindrical field falls as

$$E_\rho \propto \frac{1}{\rho}.$$

Define

$$C = \frac{V_a - V_b}{\ln(b/a)}.$$

Then

$$\mathbf{E} = \frac{C}{\rho} \hat{\rho}.$$

For the inner cylinder,

$$\hat{\mathbf{n}}_a = +\hat{\rho},$$

so

$$\sigma_a = \frac{\varepsilon_0 C}{a}.$$

For the inner face of the outer cylinder,

$$\hat{\mathbf{n}}_b = -\hat{\rho},$$

so

$$\sigma_{b,\text{inner}} = -\frac{\varepsilon_0 C}{b}.$$

Physical meaning

Charge check.

For a cylinder of length L , the facing surface areas are $2\pi aL$ and $2\pi bL$. Therefore

$$Q_a = 2\pi aL \sigma_a = 2\pi \varepsilon_0 CL,$$

while

$$Q_{b,\text{inner}} = 2\pi bL \sigma_{b,\text{inner}} = -2\pi \varepsilon_0 CL.$$

Thus

$$Q_a + Q_{b,\text{inner}} = 0.$$

The surface charge densities are different, but the total facing charges are equal and opposite.

Check / physical lesson

Useful checks

- If $V_a = V_b$, then

$$\mathbf{E} = 0.$$

- For a narrow gap $d = b - a \ll a$, write

$$\rho = a + x.$$

Since

$$\ln \frac{\rho}{a} = \ln \left(1 + \frac{x}{a} \right) \simeq \frac{x}{a}, \quad \ln \frac{b}{a} \simeq \frac{d}{a},$$

we obtain

$$\Phi \simeq V_a + \frac{V_b - V_a}{d} x,$$

which is again the parallel-plate result.

- Unlike a sphere, an infinitely long isolated cylinder does *not* have a nonconstant potential that both solves Laplace's equation and approaches zero at infinity. The logarithm does not decay:

$$\ln \rho \rightarrow \infty.$$

A second radial boundary or another reference scale is needed.

Example — which radial terms are physically allowed?

For cylindrical symmetry,

$$\Phi(\rho) = A + B \ln \rho.$$

Problem. Which terms are allowed (i) inside a charge-free cylinder containing the axis, and (ii) outside a single infinitely long cylinder?

Solution.

1. Inside a charge-free cylinder, $0 \leq \rho < a$.

Because

$$\ln \rho \rightarrow -\infty \quad (\rho \rightarrow 0),$$

regularity at the axis requires

$$B = 0.$$

Thus, if $\Phi(a) = V_a$,

$$\Phi(\rho) = V_a.$$

A nonzero logarithmic term would represent a singular line charge along the axis.

2. Separate case: outside one infinitely long cylinder.

Now take $\rho > a$ with no outer cylinder. The logarithmic term is allowed because the axis is outside the solution region.

However,

$$A + B \ln \rho$$

cannot approach zero at infinity unless $B = 0$. Therefore an isolated infinite cylinder cannot satisfy both

$$\Phi(a) = V_a \neq 0, \quad \Phi(\infty) = 0$$

within exact cylindrical symmetry.

13 The method of images: replace a difficult boundary

A conductor's unknown surface charge makes a direct Coulomb calculation difficult. The **method of images** constructs an easier calculation with imaginary point charges whose potential has the required boundary values.

An image charge is a calculation device outside the physical solution region; it is not a real charge in the vacuum.

If the constructed potential has the correct real sources, satisfies all conductor conditions, and has the required behavior far away, uniqueness makes it the physical potential in the solution region. The auxiliary continuation inside the metal has no physical meaning: the actual metal field is zero.

Method / strategy

State the physical domain first. Place images outside it, check the entire boundary and infinity, then differentiate for the field and induced surface charge.

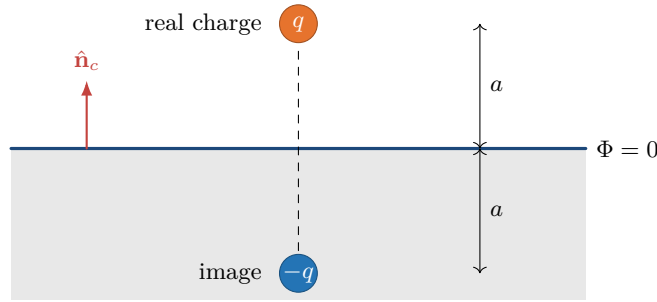
Only special geometries admit a small number of point images. A geometry without a simple image construction still has a boundary-value solution; another method may be needed.

13.1 A point charge above a grounded plane

Place q at $(0, 0, a)$ with $a > 0$. The plane $z = 0$ is grounded, the physical region is $z > 0$, and $\Phi \rightarrow 0$ at infinity. Use an image $-q$ at $(0, 0, -a)$:

$$\Phi(x, y, z) = k_e q \left[\frac{1}{\sqrt{x^2 + y^2 + (z - a)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + a)^2}} \right], \quad z > 0. \quad (13.1)$$

At every point of $z = 0$, the two distances are equal, so $\Phi = 0$. The only singularity in the physical region is the prescribed q , and the potential decays at infinity. These checks establish the solution.



The image represents the physical field only in $z > 0$.

Figure 7: Reflection in a grounded plane reverses the image sign. The physical conductor below the plane has zero field, even though the auxiliary two-charge expression would not.

13.1.1 Differentiate first, then evaluate at the boundary

Let $\varrho = \sqrt{x^2 + y^2}$. The metal normal is $\hat{\mathbf{n}}_c = \hat{\mathbf{z}}$, so

$$\sigma = -\varepsilon_0 \partial_z \Phi|_{0+}.$$

The notation 0^+ means approaching the surface from the vacuum side.

Differentiate first; only then set $z = 0$.

$$\partial_z \Phi|_{0+} = \frac{2k_e q a}{(\varrho^2 + a^2)^{3/2}}.$$

Hence

$$\sigma(\varrho) = -\frac{q a}{2\pi(\varrho^2 + a^2)^{3/2}}. \quad (13.2)$$

For $q > 0$, the induced charge is negative and most concentrated below the real charge.

Example — the total induced charge

Verify that the induced charge on the grounded plane is $-q$.

Solution. A circular ring of radius ϱ and width $d\varrho$ has area $2\pi\varrho d\varrho$. Therefore

$$Q_{\text{induced}} = \int_0^\infty \sigma(\varrho) 2\pi\varrho d\varrho = -q a \int_0^\infty \frac{\varrho d\varrho}{(\varrho^2 + a^2)^{3/2}} = -q.$$

The surface density is largest at the center, but the growing ring area also matters. Always include the correct area element.

13.1.2 Force on the real charge

A charge experiences the field of the *other* charges, so $\mathbf{F} = q\mathbf{E}_{\text{other}}$. Exclude its own singular Coulomb field.

The induced field at the real charge equals the image field at a distance $2a$:

$$\mathbf{F} = -\frac{k_e q^2}{4a^2} \hat{\mathbf{z}}. \quad (13.3)$$

The attraction occurs for either sign of q .

An independent check integrates the field of the actual induced rings:

$$F_z = k_e q \int_0^\infty \frac{a \sigma(\varrho) 2\pi\varrho d\varrho}{(\varrho^2 + a^2)^{3/2}} = -k_e q^2 a^2 \int_0^\infty \frac{\varrho d\varrho}{(\varrho^2 + a^2)^3} = -\frac{k_e q^2}{4a^2}.$$

13.1.3 Why the interaction energy has a factor of one half

The image charge reproduces the *field*, but it is not an independent physical charge. As the real charge is brought in from infinity, the conductor polarizes gradually: the induced charge, and therefore the induced potential at q , grow from zero to their final values. Since electrostatics is linear, the average induced potential during this assembly is one half of its final value. Hence

$$U = \frac{1}{2} q \Phi_{\text{induced}}(\mathbf{r}_q).$$

Here $\Phi_{\text{induced}}(\mathbf{r}_q) = -k_e q / (2a)$, because the image lies a distance $2a$ away. Therefore

$$U(a) = -\frac{k_e q^2}{4a}.$$

This also agrees with the force result: $F_a = -dU/da = -k_e q^2 / (4a^2)$. Why not use the ordinary two-charge energy $q\Phi_{\text{induced}} = -k_e q^2 / (2a)$? Because that expression would treat the image as a second independently assembled charge. It is not: the image only represents the conductor's induced charge, which is created by q itself. Counting the full pair interaction therefore counts the polarization energy twice. Week 4 develops electrostatic energy more generally. For this example, integrating the force already establishes the correct factor.

13.2 Two perpendicular grounded planes

Let the physical region be $x > 0, y > 0$, with grounded planes at $x = 0$ and $y = 0$. Place q at $(a, b, 0)$, where $a, b > 0$. A reflection across one plane alone does not satisfy both boundaries. Use the full set:

Position	Charge	Role
$(a, b, 0)$	q	Real source
$(-a, b, 0)$	$-q$	Reflection in $x = 0$
$(a, -b, 0)$	$-q$	Reflection in $y = 0$
$(-a, -b, 0)$	$+q$	Double reflection

The candidate is

$$\Phi(\mathbf{r}) = k_e \sum_{j=1}^4 \frac{q_j}{|\mathbf{r} - \mathbf{r}_j|}.$$

At $x = 0$, charges paired by x reflection cancel. At $y = 0$, charges paired by y reflection cancel. All images lie outside the domain, and the potential decays at infinity.

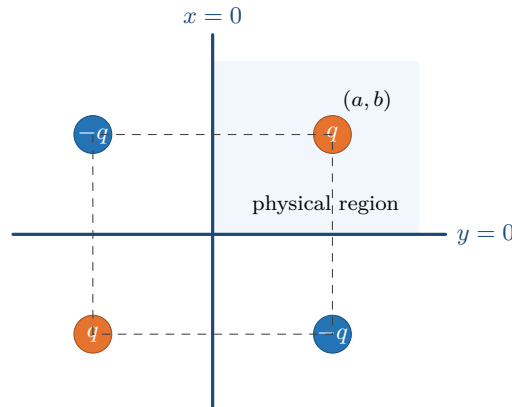


Figure 8: Three images satisfy the two perpendicular grounded planes. The shaded quadrant is the physical solution region. All four charge positions have $z = 0$.

Adding the fields of the three images at the real charge gives

$$\begin{aligned} F_x &= \frac{k_e q^2}{4} \left[-\frac{1}{a^2} + \frac{a}{(a^2 + b^2)^{3/2}} \right], \\ F_y &= \frac{k_e q^2}{4} \left[-\frac{1}{b^2} + \frac{b}{(a^2 + b^2)^{3/2}} \right], \quad F_z = 0. \end{aligned}$$

The double-reflection image repels, but each total component remains negative. For $a = b$, symmetry gives $F_x = F_y$. As $b \rightarrow \infty$, the single-plane force in the x direction is recovered.

13.3 A point charge outside a grounded sphere

A grounded sphere of radius R is centered at the origin. The real charge q is at $a\hat{\mathbf{z}}$, with $a > R$. We solve in $r > R$, require $\Phi(R, \theta) = 0$, and take $\Phi \rightarrow 0$ at infinity. Here θ is measured from the positive z axis.

Seek one image q' at $b\hat{\mathbf{z}}$ inside the sphere.

13.3.1 Derive the image from the boundary

At a point P on the sphere, the distances to the real and image charges satisfy

$$d_1^2 = R^2 + a^2 - 2Ra \cos \theta, \quad d_2^2 = R^2 + b^2 - 2Rb \cos \theta.$$

To cancel $q/d_1 + q'/d_2$ at every angle, require the constant ratio $d_2/d_1 = c = -q'/q$. Matching coefficients in $d_2^2 = c^2 d_1^2$ gives

$$b = c^2 a, \quad R^2 + b^2 = c^2(R^2 + a^2).$$

Eliminating c yields $(b - a)(ab - R^2) = 0$.

The choice $b = a$ would cancel the real source itself, destroying its required singularity. The allowed construction is

$$\boxed{b = \frac{R^2}{a}, \quad c = \frac{R}{a}, \quad q' = -q \frac{R}{a}.} \quad (13.4)$$

Therefore

$$\boxed{\Phi(r, \theta) = k_e \left[\frac{q}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2br \cos \theta}} \right], \quad r > R.} \quad (13.5)$$

The image is in the excluded metal. The constant distance ratio checks the whole spherical boundary.

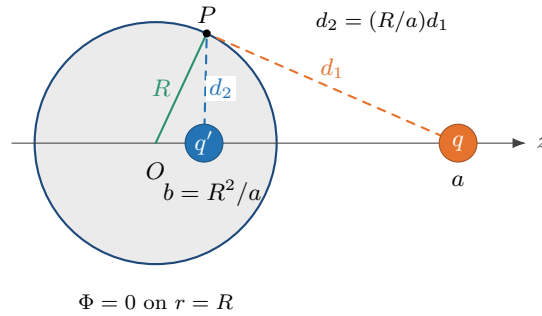


Figure 9: One image works because its distance to every boundary point is the same fixed fraction of the distance to the real charge.

13.3.2 Surface charge and its total

The metal normal is $\hat{\mathbf{r}}$. Differentiate before setting $r = R$:

$$\frac{\partial}{\partial r} \frac{1}{\sqrt{r^2 + c^2 - 2cr \cos \theta}} = -\frac{r - c \cos \theta}{(r^2 + c^2 - 2cr \cos \theta)^{3/2}}.$$

Applying this to both terms and substituting the image data gives

$$\boxed{\sigma(\theta) = -\frac{q(a^2 - R^2)}{4\pi R[a^2 + R^2 - 2aR \cos \theta]^{3/2}.} \quad (13.6)$$

For $q > 0$, the induced surface is negative everywhere and most concentrated at the nearest point, $\theta = 0$.

A ring on the sphere has area $dS = 2\pi R^2 \sin \theta d\theta$. Using

$$\int_0^\pi \frac{\sin \theta d\theta}{[a^2 + R^2 - 2aR \cos \theta]^{3/2}} = \frac{2}{a(a^2 - R^2)},$$

we obtain

$$Q_{\text{induced}} = -qR/a = q'.$$

The image reproduces the total induced charge and the exterior field; it does not describe a real point charge inside the metal.

13.3.3 Force, energy, and useful limits

The real-image separation is $(a^2 - R^2)/a$, so

$$\mathbf{F} = -\frac{k_e q^2 R a}{(a^2 - R^2)^2} \hat{\mathbf{z}}, \quad U(a) = -\frac{k_e q^2 R}{2(a^2 - R^2)}. \quad (13.7)$$

Differentiating U gives $F_z = -dU/da$.

For $a = R + h$ with $h \ll R$,

$$F_z \simeq -\frac{k_e q^2}{4h^2},$$

the plane result at a local separation h . For $a \gg R$, the leading force is $-k_e q^2 R/a^3$. The ground connection allows a net induced charge, which will matter in the next comparison.

13.4 Grounded, fixed-potential, and isolated spheres

Keep the same external charge and sphere, but now isolate the sphere and prescribe its total charge Q_c . The surface must still be an equipotential, but **its voltage is not known in advance**.

The off-center image $q' = -qR/a$ at $b = R^2/a$ is still needed: it makes the spherical surface an equipotential. However, by itself it also fixes the total charge of the conductor. To impose a different specified charge Q_c , **add a central image q_0** .

Since the potential of q_0 is $k_e q_0/r$, it has the same value everywhere on $r = R$. Thus it changes the voltage of the sphere without destroying the equipotential condition.

A Gaussian sphere with $R < r < a$ surrounds the conductor but not the real external charge. In the image construction, the enclosed charge is therefore $q' + q_0$, which must equal the prescribed conductor charge Q_c . Hence

$$q_0 = Q_c - q' = Q_c + \frac{qR}{a}, \quad V_c = \frac{k_e q_0}{R}. \quad (13.8)$$

Add $k_e q_0/r$ to Eq. (13.5). The equipotential condition and the specified total charge then uniquely determine the solution.

Example — an isolated neutral sphere is still attracted

Set the total conductor charge to zero and determine the force.

Solution. For $Q_c = 0$, the central image is $+qR/a$. It produces a repulsive contribution:

$$F_z = -\frac{k_e q^2 R a}{(a^2 - R^2)^2} + \frac{k_e q^2 R}{a^3}.$$

The first term is the attraction produced by polarization; the second comes from the central image. The total force is still attractive because $a^4 > (a^2 - R^2)^2$ for $a > R$.

At large distance, use $(1 - R^2/a^2)^{-2} = 1 + 2R^2/a^2 + \dots$:

$$F_z = -\frac{2k_e q^2 R^3}{a^5} + O\left(\frac{k_e q^2 R^5}{a^7}\right).$$

The terms proportional to a^{-3} cancel.

A neutral isolated sphere can still polarize: charge moves across its surface, creating positive and negative regions, while the total charge remains zero. A grounded sphere is different because it can also gain or lose net charge through the grounding connection. Therefore the two cases have different force laws.

For arbitrary Q_c ,

$$F_z = -\frac{k_e q^2 R a}{(a^2 - R^2)^2} + \frac{k_e q}{a^2} \left(Q_c + \frac{qR}{a} \right).$$

A sufficiently positive Q_c can repel a positive external charge.

Electrical condition	Central image q_0	Resulting quantity
Grounded: $V_c = 0$	0	$Q_c = -qR/a$
Fixed voltage: V_c	$4\pi\epsilon_0 R V_c$	$Q_c = 4\pi\epsilon_0 R V_c - qR/a$
Isolated: given Q_c	$Q_c + qR/a$	$V_c = k_e(Q_c + qR/a)/R$

The off-center image makes the sphere an equipotential; the central image adjusts the global condition: zero voltage, fixed voltage, or fixed total charge.

Problem — a conducting sphere in a uniform applied field

Find the exterior potential of a grounded sphere of radius R when $\mathbf{E} \rightarrow E_0 \hat{\mathbf{z}}$ far away.

Solution. The applied potential is $-E_0 r \cos \theta$. For no dependence on azimuthal angle, the spherical Laplacian is

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right).$$

Substitution shows that both $r \cos \theta$ and $\cos \theta / r^2$ are harmonic away from the origin. Try

$$\Phi = -E_0 r \cos \theta + \frac{A \cos \theta}{r^2}.$$

At $r = R$, zero potential for every angle requires $A = E_0 R^3$:

$$\Phi = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta, \quad \sigma(\theta) = 3\epsilon_0 E_0 \cos \theta. \quad (13.9)$$

The surface charge follows from $-\epsilon_0 \partial_r \Phi|_R$.

The induced potential decays, preserving the prescribed far field. At the poles, the normal field has magnitude $3E_0$. The total induced charge is zero because the integral of $\cos \theta \sin \theta$ from 0 to π vanishes. This solution therefore also describes an isolated neutral sphere with the same far-field reference.

Problem — a Neumann mode and its sign

In $y > 0$, seek a solution periodic in x and decaying as $y \rightarrow \infty$, with

$$\partial_{n_\Omega} \Phi(x, 0) = G \cos(kx), \quad k > 0.$$

The wavenumber k is inverse length; the spatial period, or wavelength, is $\lambda = 2\pi/k$.

Solution. Since $\hat{\mathbf{n}}_\Omega = -\hat{\mathbf{y}}$, the boundary condition is $-\Phi_y = G \cos(kx)$. Try the decaying harmonic function $Ae^{-ky} \cos(kx)$. At the boundary, $-\Phi_y = kA \cos(kx)$, so

$$\Phi(x, y) = \frac{G}{k} e^{-ky} \cos(kx). \quad (13.10)$$

The factor G/k has units of voltage. The positive and negative flux contributions cancel over one period.

The constant pattern $k = 0$ with nonzero G would require a nonzero uniform field, so it cannot satisfy decay of the potential with height.

14 Bridge to Week 4: boundary patterns become spatial modes

The disk and periodic examples suggest a systematic method: write a complicated boundary pattern as a sum of simpler patterns, solve each one, and add the results.

For a first Cartesian example, let $y > 0$ and prescribe

$$\Phi(x, 0) = V_0 \cos(kx), \quad \Phi \rightarrow 0 \quad (y \rightarrow \infty), \quad k > 0.$$

The boundary pattern repeats every $\lambda = 2\pi/k$.

Separation of variables means trying a product of functions, each depending on one coordinate:

$$\Phi(x, y) = X(x)Y(y).$$

Choose $X(x) = \cos(kx)$ to match the boundary dependence. Then $X'' = -k^2 X$, and Laplace's equation reduces to

$$Y'' - k^2 Y = 0.$$

Its general solution is $Y = Ae^{-ky} + Be^{ky}$. Decay removes B ; the boundary voltage fixes $A = V_0$:

$$\boxed{\begin{aligned}\Phi(x, y) &= V_0 e^{-ky} \cos(kx), \\ \mathbf{E} &= kV_0 e^{-ky} [\sin(kx)\hat{\mathbf{x}} + \cos(kx)\hat{\mathbf{y}}].\end{aligned}} \quad (14.1)$$

Here $\Phi_{xx} = -k^2\Phi$ and $\Phi_{yy} = +k^2\Phi$: harmonicity again comes from cancellation of curvatures.

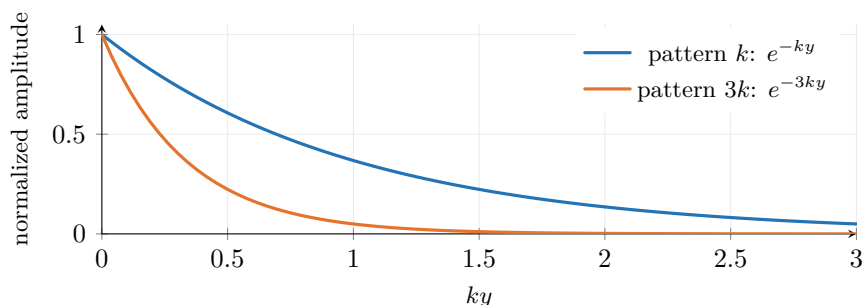


Figure 10: Shorter-wavelength boundary patterns fade more quickly with height. A pattern with wavenumber k has decay length $1/k = \lambda/(2\pi)$.

A **Fourier expansion** expresses a boundary pattern as a sum of sines and cosines. Linearity then lets us add their harmonic extensions. Short spatial features contain larger wavenumbers and influence a shallower region near the boundary.

Week 4 turns this idea into a systematic method: decompose a boundary pattern into simple spatial modes, solve each mode, and add them. The working habit remains the same: state the boundary data, choose allowed terms, determine their coefficients, and check the physical field.