

Electrodynamics & Relativity

Week 4: Building and Using the Potential

Spatial modes, capacitance, energy, and dipoles

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Main idea. Week 3 showed how boundary values select the potential. We now build a solution from simple spatial patterns, then use the field to calculate stored charge, energy, and forces. Finally, we ask which features of a charge distribution remain visible far away.

1 Continue from Week 3: add simple boundary patterns

For $y > 0$, Week 3 found the potential of a sinusoidal boundary:

$$\Phi(x, 0) = V_0 \cos(kx) \implies \Phi(x, y) = V_0 e^{-ky} \cos(kx), \quad k > 0.$$

It obeys Laplace's equation and decays with height. A **mode** is one such simple **spatial pattern**. It does not oscillate in time.

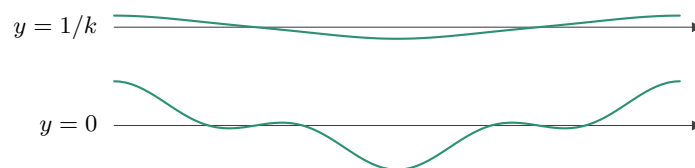
Example — a boundary with two length scales

Let $\Phi(x, 0) = V_0 [\cos(kx) + \frac{1}{2} \cos(3kx)]$, with decay at large y .

Solution. Linearity lets us add the two known solutions:

$$\Phi(x, y) = V_0 \left[e^{-ky} \cos(kx) + \frac{1}{2} e^{-3ky} \cos(3kx) \right].$$

At $y = 1/k$, the two amplitudes are $0.368V_0$ and $0.0249V_0$. The finer pattern has almost disappeared.



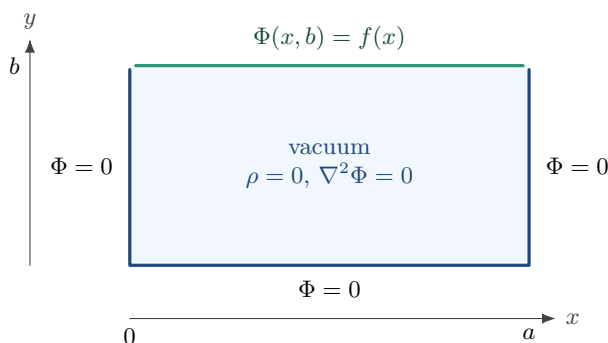
Potential profiles over one period, drawn with the same vertical scale.

2 A rectangle: state the problem first

Consider the vacuum cross-section $0 < x < a$, $0 < y < b$ of a very long rectangular channel. Nothing varies along z , so $\Phi = \Phi(x, y)$. Three walls are grounded; the top has prescribed potential $f(x)$:

$$\begin{aligned} \Phi(0, y) = \Phi(a, y) = 0, \quad \Phi(x, 0) = 0, \\ \Phi(x, b) = f(x), \quad \Phi_{xx} + \Phi_{yy} = 0 \quad \text{inside.} \end{aligned} \quad (2.1)$$

A varying $f(x)$ requires separately controlled electrode strips. One connected metal wall can have only a constant potential.



Small gaps at the top corners keep electrodes at different voltages apart.

2.1 Turn one equation in two variables into two ordinary equations

Separation of variables begins by trying one product, $\Phi = X(x)Y(y)$. This is a building block; sums of products will give more general boundary patterns.

Separate the coordinate dependence

Substitution into Laplace's equation gives

$$X''Y + XY'' = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y}.$$

The left side depends only on x , the right only on y . Holding y fixed and varying x shows that both must be the same constant. For the grounded side walls we choose $-k^2$:

$$X'' + k^2X = 0, \quad Y'' - k^2Y = 0.$$

Thus X contains sines and cosines, while Y contains exponentials. The next page applies the walls and shows why $k^2 > 0$ is the useful choice.

Method / strategy

Use the three zero-voltage walls to choose the allowed functions first. Then use the remaining wall to determine how much of each function is needed.

2.2 Apply the three grounded walls, one at a time

Write the separated functions as

$$X = A \sin(kx) + B \cos(kx), \quad Y = C \sinh(ky) + D \cosh(ky).$$

Here $\sinh u = (e^u - e^{-u})/2$ and $\cosh u = (e^u + e^{-u})/2$. In particular, $\sinh 0 = 0$ and $\cosh 0 = 1$.

The walls select the modes

Left wall: $X(0) = 0$ gives $B = 0$.

Right wall: $X(a) = 0$ gives $\sin(ka) = 0$. A nonzero mode requires

$$k = k_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \dots$$

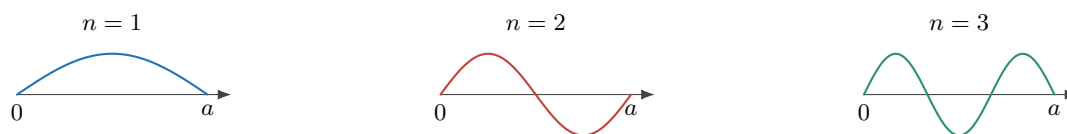
Bottom wall: $Y(0) = 0$ gives $D = 0$.

The side conditions require $X(0) = X(a) = 0$. If the separation constant is zero, then $X = A + Bx$, and the only linear function vanishing at both ends is $X \equiv 0$. If the separation constant is $+k^2$, with $k > 0$, then X is hyperbolic and again cannot vanish at two distinct points unless $X \equiv 0$. Therefore we choose the remaining sign, $-k^2$, which gives $X'' + k^2X = 0$, and hence nontrivial sine modes satisfying both boundary conditions.

Normalize each y -factor to equal one at the top:

$$\Phi_n(x, y) = F_n \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh(n\pi y/a)}{\sinh(n\pi b/a)}. \quad (2.2)$$

The coefficient F_n is a voltage: the amplitude of mode n on the top wall.



The top-wall patterns fit one, two, or three half-waves between the grounded sides.

Example — a top voltage that needs only one mode

Let $f(x) = V_0 \sin(\pi x/a)$. Find the interior potential.

Solution. Only $n = 1$ is needed, with $F_1 = V_0$:

$$\Phi = V_0 \sin(\pi x/a) \frac{\sinh(\pi y/a)}{\sinh(\pi b/a)}.$$

Check. The sine vanishes on both sides, the numerator vanishes at the bottom, and the ratio equals one at the top. Also $\Phi_{xx} = -(\pi/a)^2 \Phi$ and $\Phi_{yy} = +(\pi/a)^2 \Phi$, so Laplace's equation holds.

Physical meaning

The boundary selects a spatial wavelength $\lambda_n = 2a/n$. Higher n means finer variation along the top wall, not a higher oscillation frequency in time.

3 Fourier coefficients: how much of each mode?

For a general top voltage, add the allowed modes:

$$\Phi(x, y) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh(n\pi y/a)}{\sinh(n\pi b/a)}. \quad (3.1)$$

At $y = b$, this becomes a **Fourier sine series** for the given function:

$$f(x) = \sum_{n=1}^{\infty} F_n \sin(n\pi x/a), \quad 0 < x < a.$$

To extract one coefficient, multiply by its sine and integrate. Different modes cancel under this operation. This is **orthogonality**:

$$\int_0^a \sin(n\pi x/a) \sin(m\pi x/a) dx = \begin{cases} 0, & n \neq m, \\ a/2, & n = m. \end{cases}$$

An integral that selects just one coefficient

Using $2 \sin u \sin v = \cos(u - v) - \cos(u + v)$ gives the relation above. Multiplying the boundary series by $\sin(m\pi x/a)$ then gives

$$\int_0^a f(x) \sin(m\pi x/a) dx = \frac{a}{2} F_m.$$

Hence, relabeling m as n ,

$$F_n = \frac{2}{a} \int_0^a f(x) \sin(n\pi x/a) dx. \quad (3.2)$$

The integral has units of voltage times length; $2/a$ leaves a voltage.

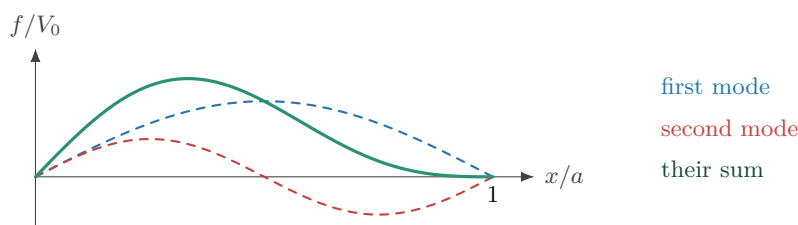
Example — read the coefficients without integrating

Suppose

$$f(x) = V_0 \left[\sin(\pi x/a) + \frac{1}{2} \sin(2\pi x/a) \right].$$

Solution. The coefficients are already visible: $F_1 = V_0$, $F_2 = V_0/2$, and all the others are zero. Therefore

$$\Phi = V_0 \sin(\pi x/a) \frac{\sinh(\pi y/a)}{\sinh(\pi b/a)} + \frac{V_0}{2} \sin(2\pi x/a) \frac{\sinh(2\pi y/a)}{\sinh(2\pi b/a)}.$$



The second mode makes the left half higher and the right half lower.

4 A constant top voltage: a complete solution

Keep the three grounded walls and hold the top at $V_0 > 0$. The top electrode is separated from the side walls by narrow insulating gaps. We use the ideal boundary value $f(x) = V_0$ for $0 < x < a$.

Example — find the coefficients and the potential

Solution. Equation (3.2) gives

$$F_n = \frac{2V_0}{a} \int_0^a \sin(n\pi x/a) dx = \frac{2V_0}{n\pi} [1 - (-1)^n].$$

Thus $F_n = 0$ for even n and $F_n = 4V_0/(n\pi)$ for odd n . Substitution into Eq. (3.1) gives

$$\Phi(x, y) = \frac{4V_0}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin(n\pi x/a)}{n} \frac{\sinh(n\pi y/a)}{\sinh(n\pi b/a)}. \quad (4.1)$$

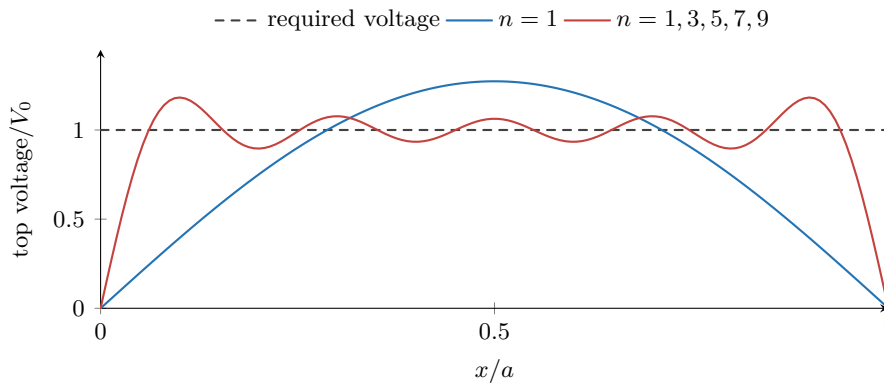
Every term satisfies Laplace's equation and the three grounded walls. The Fourier coefficients supply the fourth wall. Uniqueness then identifies this as the physical interior potential.

4.1 Why do the even modes disappear?

The boundary is unchanged by reflection about $x = a/2$. But

$$\sin\left(\frac{n\pi(a-x)}{a}\right) = (-1)^{n+1} \sin\left(\frac{n\pi x}{a}\right).$$

Odd n gives the same sign on reflection; even n gives the opposite sign. Only the symmetric, odd- n patterns can occur here.



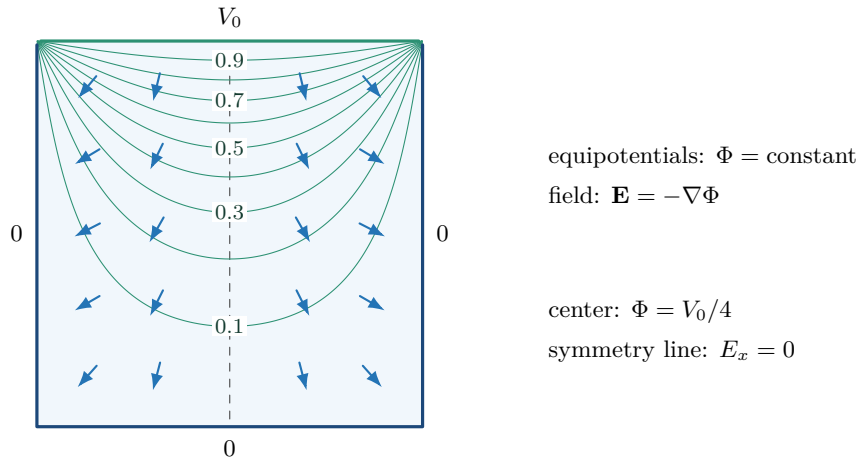
Adding modes reproduces the flat boundary away from the corners. The narrow oscillations near each end come from truncating the series.

Physical meaning

The ideal top and side voltages disagree at the two corner points. The series matches the open edges, and the interior is smooth. Its behavior at an ideal corner cannot describe a finite electrode gap.

5 Read the solution: field direction and penetration

Take a square, $b = a$. The figure evaluates Eq. (4.1), showing equal-voltage curves and field directions.



Square channel with the top at V_0 and the other walls at zero. Green curves give Φ/V_0 ; blue arrows show the direction of $\mathbf{E}$. Arrow lengths are fixed so that direction remains clear.

The field points from the top electrode toward the grounded walls, perpendicular to the equipotentials. Reflection symmetry gives

$$\Phi(a - x, y) = \Phi(x, y) \implies E_x(a/2, y) = 0.$$

Away from the driven wall, fine modes are suppressed. At a depth $s = b - y$, when $k_n y$ and $k_n b$ are large,

$$\frac{\sinh(k_n y)}{\sinh(k_n b)} \simeq e^{-k_n s}, \quad k_n = \frac{n\pi}{a}.$$

This is the same decay seen at the end of Week 3. The exact hyperbolic ratio remains valid near the bottom wall.

Example — the center voltage, checked without a series

What is $\Phi(a/2, a/2)$ for the square?

Solution. Keep only $n = 1$ as a first approximation:

$$\frac{\Phi_1(a/2, a/2)}{V_0} = \frac{4}{\pi} \frac{\sinh(\pi/2)}{\sinh \pi} = \frac{2}{\pi \cosh(\pi/2)} \simeq 0.2537.$$

For an independent exact answer, consider four separate problems, each with one wall at V_0 and three at zero. Rotation makes their center voltages equal. Adding them puts all four walls at V_0 , whose interior solution is the constant V_0 . Therefore

$$\Phi(a/2, a/2) = V_0/4.$$

The first mode is already within 1.5% at the center.

Check / physical lesson

For $V_0 > 0$, the full solution obeys $0 < \Phi < V_0$ in the interior. Boundary values, symmetry, and the mean-value property provide checks before any field derivative is calculated.

6 Capacitance: charge stored per volt

A capacitor consists of two conductors carrying $+Q$ and $-Q$. Let $\Delta V = V_+ - V_- > 0$ be their potential difference. For fixed geometry in vacuum, doubling ΔV doubles the field and the surface charge. Their ratio is the **capacitance**:

$$C = \frac{Q}{\Delta V}, \quad Q = C\Delta V. \quad (6.1)$$

The SI unit is the farad: $1 \text{ F} = 1 \text{ C/V}$. The charge Q is the magnitude on *one* conductor; the sum of the two conductor charges is zero.

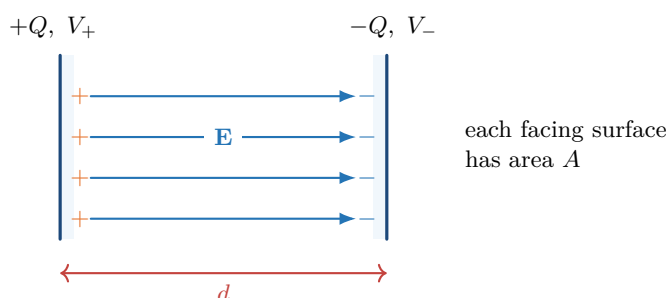
6.1 Parallel plates: reuse the field from Week 3

For plate area A and separation d much smaller than the lateral dimensions, neglect the edge field. Then

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}, \quad \Delta V = Ed = \frac{Qd}{\epsilon_0 A}.$$

Dividing Q by ΔV gives (is determined solely by the **conductor geometry** and the **permittivity of the medium**.)

$$C_{\text{plates}} = \frac{\epsilon_0 A}{d}. \quad (6.2)$$



A parallel-plate capacitor with charges on the **facing surfaces** of the two electrodes. **In electrostatic equilibrium the field inside each metal electrode is zero**, so the potential drop occurs across the vacuum gap of width d . Neglecting edge effects, the equal and opposite facing charges $\pm Q$ produce an approximately uniform field between the plates.

Example — a capacitor we will use again

Two square plates have area $A = 100 \text{ cm}^2 = 10^{-2} \text{ m}^2$ and separation $d = 1.00 \text{ mm}$. Find C , Q , and E at $\Delta V = 100 \text{ V}$.

Solution. With $\epsilon_0 \simeq 8.85 \times 10^{-12} \text{ F/m}$,

$$\begin{aligned} C &= 8.85 \times 10^{-11} \text{ F} = 88.5 \text{ pF}, \\ Q &= C\Delta V = 8.85 \text{ nC}, \\ E &= \Delta V/d = 1.00 \times 10^5 \text{ V/m}. \end{aligned}$$

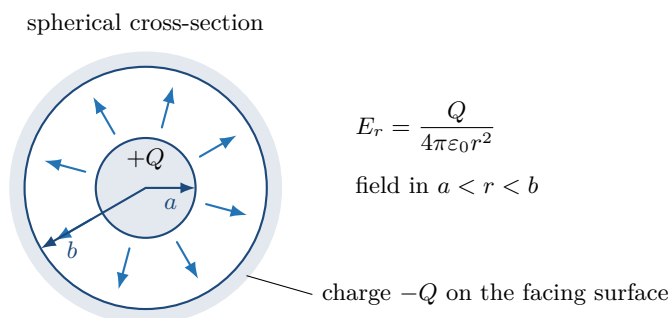
Check. Doubling the voltage doubles Q and E , while C stays the same. Doubling the area doubles C ; doubling the gap halves it.

Physical meaning

A larger capacitance means more charge can be stored at the same voltage difference. It does not mean the capacitor is already charged.

7 Concentric spheres: read capacitance from the field

The inner conductor has radius a and charge $+Q$. The inner radius of the enclosing conductor is $b > a$; that conductor has charge $-Q$. With no external sources, the field is confined to the vacuum gap.



Example — turn the known radial field into C

Find the capacitance of the two spheres.

Solution. Week 3 gave the radial field. Integrate it across the gap:

$$\Delta V = V_a - V_b = \int_a^b E_r dr = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right).$$

The sign is positive because the inner sphere has the higher voltage. Hence

$$C_{\text{spheres}} = 4\pi\epsilon_0 \frac{ab}{b-a}. \quad (7.1)$$

For $a = 5.0$ cm and $b = 10.0$ cm,

$$C = 4\pi\epsilon_0(0.100 \text{ m}) \simeq 11.1 \text{ pF}.$$

7.1 Two limits explain the result

An isolated sphere. Moving the outer conductor to infinity gives

$$C_{\text{isolated sphere}} = 4\pi\epsilon_0 a, \quad \Phi(\infty) = 0. \quad (7.2)$$

Here the voltage is measured relative to infinity. For the same $a = 5.0$ cm, $C \simeq 5.56$ pF.

A narrow gap. Set $b = a + d$ with $d \ll a$. Then

$$C = 4\pi\epsilon_0 \frac{a(a+d)}{d} \simeq \frac{\epsilon_0(4\pi a^2)}{d}.$$

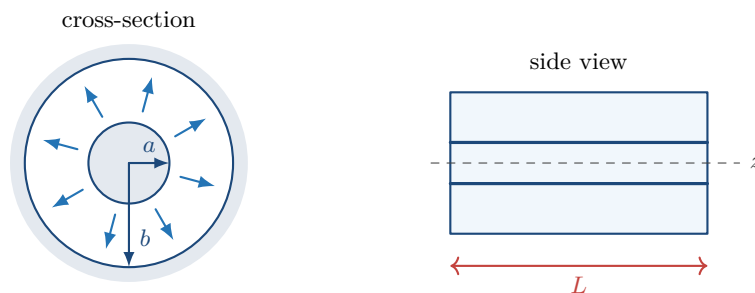
This is the parallel-plate formula with area $4\pi a^2$.

Check / physical lesson

The result has units $(\text{F/m}) \times \text{m} = \text{F}$. At fixed a , bringing the outer conductor closer increases C : less voltage is needed to hold the same Q .

8 Coaxial cylinders: capacitance per length

Two long concentric cylinders have radii $a < b$ and length L . Ignore end effects. The inner and outer conductors carry charges $+Q$ and $-Q$, with $\lambda = Q/L$. As in Week 3, use ϱ for distance from the axis.



The cross-section sets the field; extending the length adds more identical sections.

Example — reuse the cylindrical field

Find C/L for the coaxial capacitor.

Solution. The Week 3 field, or Gauss's law on a cylinder, gives

$$E_{\varrho} = \frac{\lambda}{2\pi\epsilon_0\varrho}.$$

Therefore

$$\Delta V = \int_a^b E_{\varrho} d\varrho = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right).$$

Since $Q = \lambda L$,

$$C_{\text{coax}} = \frac{2\pi\epsilon_0 L}{\ln(b/a)}, \quad \frac{C}{L} = \frac{2\pi\epsilon_0}{\ln(b/a)}. \quad (8.1)$$

For $b = 2a$ in vacuum,

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln 2} \simeq 80.2 \text{ pF/m}.$$

A 2.0 m length therefore has $C \simeq 160 \text{ pF}$.

Check / physical lesson

The logarithm acts on the dimensionless ratio b/a , not on a length. Doubling L doubles C . Scaling both radii by the same factor leaves C/L unchanged because their ratio is unchanged.

8.1 The same narrow-gap check

For $b = a + d$ and $d \ll a$, use $\ln(1 + d/a) \simeq d/a$:

$$C \simeq \frac{\epsilon_0(2\pi aL)}{d}.$$

The area is now $2\pi aL$, again giving the parallel-plate form. **The three capacitances follow from the same operation: integrate the known field to obtain ΔV , then form $Q/\Delta V$.**

9 Energy: the work needed to charge a capacitor

Start with an uncharged capacitor of fixed geometry. Transfer charge slowly from one conductor to the other. When the charges are $+q$ and $-q$, their potential difference is $\Delta V(q) = q/C$.

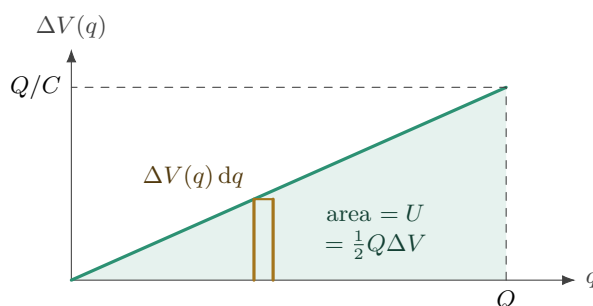
Moving the next small charge dq from the lower to the higher potential requires work

$$dU = \Delta V(q) dq = \frac{q}{C} dq.$$

The early charge is easy to transfer; later charge must be moved against a larger voltage. Integrating from $q = 0$ to $q = Q$ gives

$$U = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} = \frac{1}{2}Q\Delta V = \frac{1}{2}C(\Delta V)^2. \quad (9.1)$$

These are three forms of the same stored energy. Choose the one suited to the quantities given in the problem.



The work is the area under the charging curve. Its triangular shape gives the factor $1/2$.

Example — stored energy and doubling the voltage

Use the 88.5 pF capacitor from Section 6, charged to 100 V.

Solution. Its stored energy is

$$U = \frac{1}{2}(88.5 \times 10^{-12})(100)^2 = 4.43 \times 10^{-7} \text{ J} = 0.443 \mu\text{J}.$$

If the voltage is doubled at fixed geometry, the charge doubles and the stored energy becomes $4U = 1.77 \mu\text{J}$.

Check. A volt times a coulomb is a joule, so each form in Eq. (9.1) has energy units.

Physical meaning

Once charged and disconnected, an ideal capacitor retains its charge and stored energy. The energy can later be released by allowing charge to move through an external circuit.

10 The same energy is stored in the electric field

For a vacuum capacitor, the energy can also be calculated from the field throughout space:

$$U = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 dV, \quad u_E = \frac{\varepsilon_0 E^2}{2}. \quad (10.1)$$

The **energy density** u_E is energy per volume, measured in J/m³. Here $E^2 = \mathbf{E} \cdot \mathbf{E}$.

From charge energy to field energy

For a localized charge distribution with finite self-energy, assembly gives $U = \frac{1}{2} \int \rho \Phi dV$, with $\Phi(\infty) = 0$. Surface charges are included by the corresponding surface integrals. Using Gauss's law and $\nabla \Phi = -\mathbf{E}$,

$$\nabla \cdot (\Phi \mathbf{E}) = -E^2 + \frac{\rho \Phi}{\varepsilon_0}.$$

Integrate and apply the divergence theorem:

$$U = \frac{\varepsilon_0}{2} \int E^2 dV + \frac{\varepsilon_0}{2} \oint \Phi \mathbf{E} \cdot \hat{\mathbf{n}} dS.$$

On a sphere taken to infinity, the last term vanishes for localized sources, giving Eq. (10.1). Ideal point charges have divergent self-energies; their finite pair interaction energies require excluding those self-terms.

Example — check the plate energy directly

Neglecting edges, the uniform field occupies volume Ad and $E = \Delta V/d$. Thus

$$U = \frac{\varepsilon_0 E^2}{2} Ad = \frac{1}{2} \left(\frac{\varepsilon_0 A}{d} \right) (\Delta V)^2 = \frac{1}{2} C (\Delta V)^2.$$

This independently reproduces the charging calculation.

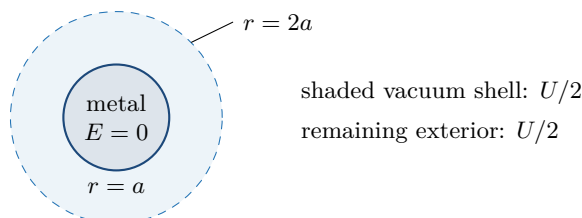
Example — where is the energy of a charged sphere?

An isolated metal sphere of radius a carries Q . Its field is zero in the metal and $E = k_e Q/r^2$ outside, where $k_e = 1/(4\pi\varepsilon_0)$.

Solution. A spherical shell has volume $4\pi r^2 dr$, so

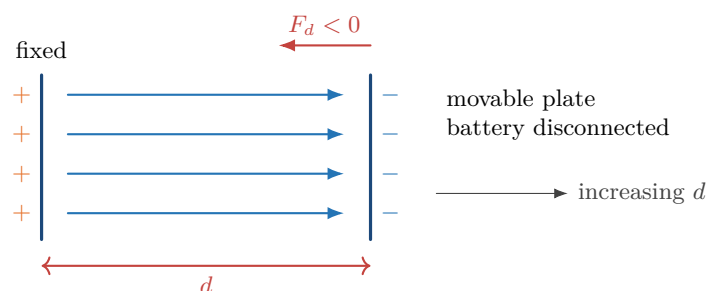
$$U = \frac{\varepsilon_0}{2} \int_a^\infty \left(\frac{Q}{4\pi\varepsilon_0 r^2} \right)^2 4\pi r^2 dr = \left[\frac{Q^2}{8\pi\varepsilon_0 a} \right] = \frac{Q^2}{2C}.$$

Replacing the upper limit by $2a$ gives $U/2$: half of the total field energy lies between a and $2a$.



11 Plate attraction: force from stored energy

Charge a parallel-plate capacitor and **disconnect it**. The charges are now fixed at $+Q$ and $-Q$. Let d increase when the right plate moves to the right; keep the left plate fixed and ignore edge effects.



Example — differentiate while keeping Q fixed

Find the force on the movable plate.

Solution. Since $C = \epsilon_0 A/d$,

$$U(d) = \frac{Q^2}{2C} = \frac{Q^2 d}{2\epsilon_0 A}.$$

The electric force is minus the rate at which energy increases with separation:

$$F_d = - \left(\frac{\partial U}{\partial d} \right)_Q = - \frac{Q^2}{2\epsilon_0 A}. \quad (11.1)$$

The negative sign means attraction. Since $E = Q/(\epsilon_0 A)$,

$$|F_d| = \frac{1}{2} \epsilon_0 E^2 A, \quad \frac{|F_d|}{A} = \frac{1}{2} \epsilon_0 E^2. \quad (11.2)$$

For our numerical capacitor, $|F_d| = 4.43 \times 10^{-4} \text{ N} = 0.443 \text{ mN}$ at its original separation.

Check / physical lesson

Independent force check. One uniform charged sheet produces field $\sigma/(2\epsilon_0)$ on either side. The force on a plate is due to the *other* plate:

$$|F_d| = QE_{\text{other}} = Q \frac{\sigma}{2\epsilon_0} = \frac{Q^2}{2\epsilon_0 A}.$$

Using the full gap field σ/ϵ_0 would also count the plate's own field and give twice the correct force.

Physical meaning

The attraction tends to reduce the field-filled volume and the stored energy when Q is fixed. The pressure has the same numerical value as u_E ; indeed $1 \text{ J/m}^3 = 1 \text{ N/m}^2$.

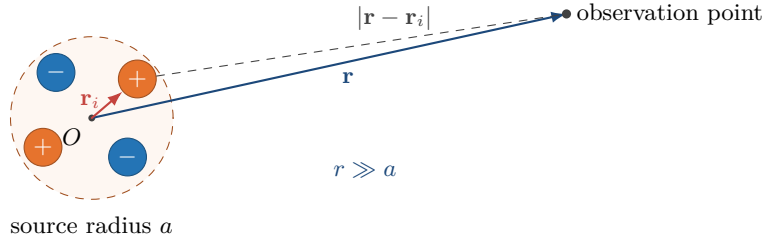
If the battery stays connected, ΔV is fixed and charge can flow. The battery exchanges energy with the capacitor, so differentiating $\frac{1}{2}C(\Delta V)^2$ alone is insufficient. The attraction still obeys Eq. (11.2); page 17 gives the optional energy calculation.

12 Far from the charges: keep the leading terms

For charges q_i at positions $\mathbf{r}_i$, the exact vacuum potential is

$$\Phi(\mathbf{r}) = k_e \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}, \quad \Phi(\infty) = 0.$$

The observation point is $\mathbf{r} = r\hat{\mathbf{r}}$; $\hat{\mathbf{r}}$ is a unit vector pointing from the origin toward that point. Suppose every charge lies within a distance a of the origin and the observer is far away, $r \gg a$.



Expand the distance, not the charge distribution

Factor out the large distance r :

$$\frac{1}{|\mathbf{r} - \mathbf{r}_i|} = \frac{1}{r} \left(1 - \frac{2\hat{\mathbf{r}} \cdot \mathbf{r}_i}{r} + \frac{r_i^2}{r^2} \right)^{-1/2}.$$

Using $(1 + u)^{-1/2} = 1 - u/2 + \dots$, keep terms through first order in r_i/r :

$$\frac{1}{|\mathbf{r} - \mathbf{r}_i|} = \frac{1}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}_i}{r^2} + O\left(\frac{a^2}{r^3}\right).$$

The next term is quadratic in the source coordinates and falls as $1/r^3$.

Key relations

The first two **multipole moments** are the total charge and dipole moment:

$$Q = \sum_i q_i, \quad \mathbf{p} = \sum_i q_i \mathbf{r}_i. \quad (12.1)$$

They give the first terms of the distant potential:

$$\Phi(\mathbf{r}) = k_e \left[\frac{Q}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + \dots \right]. \quad (12.2)$$

For continuous charge, replace the sums by

$$Q = \int \rho(\mathbf{r}') dV', \quad \mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') dV'.$$

Net charge gives the monopole term. If $Q = 0$, that term cancels; a nonzero dipole moment then determines the leading far field. The expansion converges outside the source sphere; keeping only a few terms is useful when a/r is small.

13 Calculate the moments before calculating the far field

Choose the origin and draw the charge positions. In Cartesian components, $p_x = \sum_i q_i x_i$, with analogous formulas for p_y and p_z . The dipole moment has units of charge times length, C m.

Example — a neutral pair has a dipole moment

Put $+q$ at $z = d/2$ and $-q$ at $z = -d/2$.

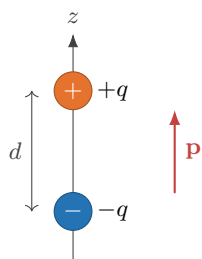
Solution. The total charge vanishes, but the two contributions to p_z add:

$$Q = q - q = 0, \quad p_z = q \frac{d}{2} + (-q) \left(-\frac{d}{2} \right) = qd.$$

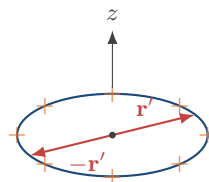
Thus

$$\mathbf{p} = qd \hat{\mathbf{z}}, \quad \Phi_{\text{far}} = k_e \frac{p \cos \theta}{r^2}, \quad r \gg d, \quad (13.1)$$

where θ is measured from the positive z axis. The dipole moment points from the negative charge to the positive one. For $q = 2.0 \text{ nC}$ and $d = 4.0 \text{ mm}$, $p = 8.0 \times 10^{-12} \text{ C m}$.



opposite signs: contributions add



equal signs: contributions cancel

Example — a uniformly charged ring has no centered dipole

A ring of radius a lies in the xy plane and carries total charge Q . Choose the origin at its center.

Solution. Every element dq at $\mathbf{r}'$ has an equal partner at $-\mathbf{r}'$. Their dipole contributions cancel:

$$\mathbf{r}' dq + (-\mathbf{r}') dq = 0 \quad \implies \quad \boxed{\mathbf{p} = 0.}$$

For $Q \neq 0$, the leading potential is still $k_e Q/r$. The first correction is quadrupolar, of order $1/r^3$; page 20 checks this directly. Uniform charge does not imply zero total charge.

The origin matters when $Q \neq 0$

Moving the origin to the old position $\mathbf{a}_0$ changes every coordinate to $\mathbf{r}_i - \mathbf{a}_0$. Hence $\mathbf{p}_{\text{new}} = \mathbf{p}_{\text{old}} - Q\mathbf{a}_0$. A neutral system's dipole moment is independent of the origin. For the charged ring, $\mathbf{p} = 0$ specifically refers to its center.

14 The dipole field: differentiate the potential

For $\mathbf{p} = p\hat{\mathbf{z}}$, use $\Phi = k_e p \cos \theta / r^2$. The spherical components of the gradient give

$$E_r = -\frac{\partial \Phi}{\partial r}, \quad E_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta}.$$

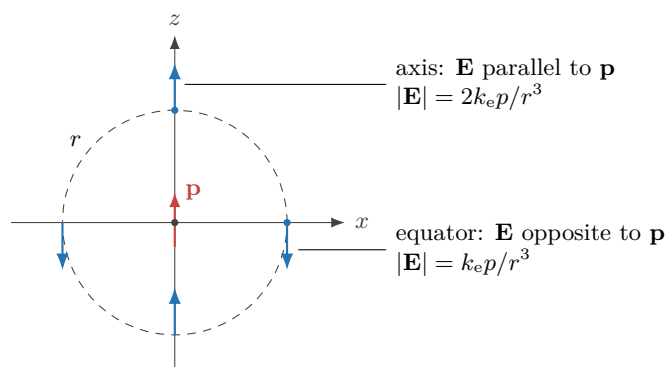
The factor $1/r$ converts an angular change $d\theta$ into the distance $r d\theta$ traveled along an arc. Thus

$$E_r = \frac{2k_e p \cos \theta}{r^3}, \quad E_\theta = \frac{k_e p \sin \theta}{r^3}, \quad E_\phi = 0. \quad (14.1)$$

The equivalent vector form, for any orientation of $\mathbf{p}$, is

$$\mathbf{E}_{\text{dip}} = \frac{k_e}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]. \quad (14.2)$$

For a finite charge pair these are far-field formulas, $r \gg d$. They are exact for an ideal point dipole away from its location.



Field directions at four points the same distance from a dipole. The arrows indicate direction; the labels specify magnitude.

Example — compare the axis and the equator

For the pair on the previous page, $p = 8.0 \times 10^{-12} \text{ C m}$. Find the distant field at $r = 0.10 \text{ m}$.

Solution. With $k_e \simeq 8.99 \times 10^9 \text{ N m}^2/\text{C}^2$,

$$\mathbf{E}_{\text{axis}} \simeq 144 \hat{\mathbf{z}} \text{ V/m}, \quad \mathbf{E}_{\text{equator}} \simeq -71.9 \hat{\mathbf{z}} \text{ V/m}.$$

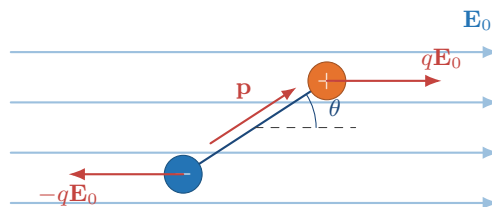
On either side of the dipole axis the field points along $\mathbf{p}$; on the equatorial plane it points the other way.

Check / physical lesson

On the equator, $\Phi = 0$ but $\mathbf{E} \neq 0$: the angular slope of Φ is nonzero. A dipole potential falls as $1/r^2$ and its field as $1/r^3$, one power faster than their monopole counterparts.

15 A dipole in an applied field: rotation and energy

Place a small rigid dipole in a uniform external field $\mathbf{E}_0$. Keep its moment $\mathbf{p} = q\mathbf{d}$ fixed in magnitude. The vector $\mathbf{d}$ points from $-q$ to $+q$. The two forces are $+q\mathbf{E}_0$ and $-q\mathbf{E}_0$. They add to zero but act at different positions.



opposite forces, different lines of action

Torque and energy of a fixed dipole

Taking the midpoint as origin, the two torques add:

$$\boldsymbol{\tau} = \frac{\mathbf{d}}{2} \times (q\mathbf{E}_0) + \left(-\frac{\mathbf{d}}{2}\right) \times (-q\mathbf{E}_0) = \boxed{\mathbf{p} \times \mathbf{E}_0}.$$

The interaction energy in the external potential is

$$U_{\text{ext}} = q\Phi_{\text{ext}}(\mathbf{r}_+) - q\Phi_{\text{ext}}(\mathbf{r}_-) = -q\mathbf{E}_0 \cdot \mathbf{d} = \boxed{-\mathbf{p} \cdot \mathbf{E}_0}.$$

The dipole's internal energy is unchanged as it rotates and is omitted. Here the applied field is prescribed independently; this is not the energy of charging a capacitor.

Thus $\tau = pE_0 \sin \theta$ and $U_{\text{ext}} = -pE_0 \cos \theta$. Parallel alignment is the energy minimum and is stable. Antiparallel alignment is an energy maximum: its torque is zero exactly at $\theta = \pi$, but a small rotation moves it farther away.

Example — release a dipole from 60°

Let $p = 8.0 \times 10^{-12} \text{ C m}$ and $E_0 = 1.0 \times 10^5 \text{ V/m}$. Find the initial torque and the energy released on reaching parallel alignment.

Solution.

$$\tau = pE_0 \sin 60^\circ = 6.93 \times 10^{-7} \text{ N m},$$

$$U(60^\circ) - U(0) = pE_0(1 - \cos 60^\circ) = \boxed{4.00 \times 10^{-7} \text{ J}}.$$

The released energy becomes rotational kinetic energy or is dissipated if the motion is damped.

Check / physical lesson

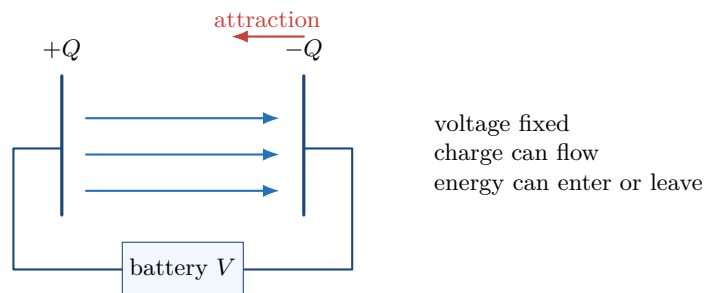
A neutral dipole in a *uniform* field has zero net force and can have nonzero torque. If the applied field varies across the dipole, the two forces need not cancel and translation is also possible.

OPTIONAL — A capacitor connected to a battery

Enrichment only. This material is not examined.

16 Fixed voltage: include the battery's energy

Keep the capacitor connected to an ideal battery holding $V = \Delta V$ constant. As the separation changes, $C(d)$ and $Q = C(d)V$ change. The field energy is $U_{\text{field}} = \frac{1}{2}C(d)V^2$.

**What energy must be differentiated?**

When the battery supplies charge dQ , its energy changes by $dU_{\text{bat}} = -V dQ$. Since V is fixed,

$$d(U_{\text{field}} + U_{\text{bat}}) = d(U_{\text{field}} - QV).$$

Up to an irrelevant constant, the energy for the mechanical calculation is

$$\mathcal{G} = U_{\text{field}} - QV = -\frac{1}{2}C(d)V^2.$$

Consequently,

$$F_d = - \left(\frac{\partial \mathcal{G}}{\partial d} \right)_V = \frac{V^2}{2} \frac{dC}{dd} = -\frac{\varepsilon_0 A V^2}{2d^2}.$$

The force is attractive and agrees with $-|F_d| = -\varepsilon_0 E^2 A/2$, using $E = V/d$.

Example — pull the plates from d_0 to $2d_0$

Let $C_0 = \varepsilon_0 A/d_0$, $Q_0 = C_0 V$, and $U_0 = C_0 V^2/2$.

Solution. At the final separation,

$$C_1 = C_0/2, \quad Q_1 = Q_0/2, \quad U_{\text{field},1} = U_0/2.$$

Charge $Q_0/2$ returns to the battery, increasing its energy by $VQ_0/2 = U_0$. The combined energy therefore rises by $-U_0/2 + U_0 = U_0/2$. This equals the external work against attraction:

$$W_{\text{ext}} = \int_{d_0}^{2d_0} \frac{\varepsilon_0 A V^2}{2d^2} dd = U_0/2.$$

Check / physical lesson

Fixed Q : differentiate $Q^2/(2C)$. Fixed V : include the battery and differentiate $-CV^2/2$. Both give the same force at the same instantaneous Q , V , and geometry.

OPTIONAL — Circular boundary patterns

Enrichment only. This material is not examined.

17 A circle: the angular pattern selects the radial power

Consider a charge-free disk $0 \leq \varrho < a$, with no z dependence. The potential around its boundary is prescribed by separate electrodes; it is not the surface of one connected conductor if the voltage varies. The polar angle is ϕ and $x = \varrho \cos \phi$, $y = \varrho \sin \phi$.

The circular version of separation

In these coordinates,

$$\nabla^2 \Phi = \frac{1}{\varrho} \partial_\phi (\varrho \partial_\phi \Phi) + \frac{1}{\varrho^2} \partial_\varrho^2 \Phi = 0.$$

For a mode $\Phi = R(\varrho) \cos(n\phi)$, substitution gives

$$\varrho^2 R'' + \varrho R' - n^2 R = 0.$$

Trying $R = \varrho^s$ gives $s^2 = n^2$, so the radial powers are ϱ^n and ϱ^{-n} . Integer n makes the pattern repeat after one full turn. Regularity at the axis removes ϱ^{-n} . The $n = 0$ case gives a constant and $\ln \varrho$; regularity removes the logarithm.

Example — two angular modes make two different fields

Prescribe $\Phi(a, \phi) = V_1 \cos \phi + V_2 \cos(2\phi)$.

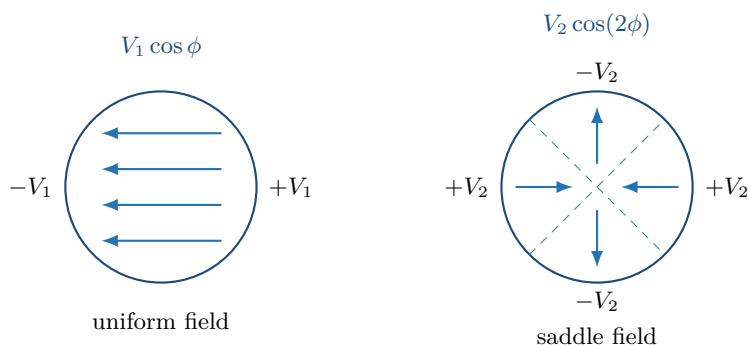
Solution. Extend each mode with its allowed radial power:

$$\Phi = V_1 \frac{\varrho}{a} \cos \phi + V_2 \frac{\varrho^2}{a^2} \cos(2\phi) = \frac{V_1}{a} x + \frac{V_2}{a^2} (x^2 - y^2).$$

Therefore

$$\mathbf{E} = - \left(\frac{V_1}{a} + \frac{2V_2 x}{a^2} \right) \hat{\mathbf{x}} + \frac{2V_2 y}{a^2} \hat{\mathbf{y}}.$$

The first mode gives a uniform field; the second gives the saddle potential encountered in Week 3.



Each mode shown separately for positive amplitude. The dashed diagonals in the right panel have $\Phi = 0$.

Check / physical lesson

At $\varrho = a$, both radial factors become one. In Cartesian form, $\partial_x^2 \Phi = 2V_2/a^2$ and $\partial_y^2 \Phi = -2V_2/a^2$, so the curvatures cancel. At $\varrho = a/2$, the two potential amplitudes are reduced by $1/2$ and $1/4$ respectively.

OPTIONAL — Spherical modes and the link to dipoles

Enrichment only. This material is not examined.

18 A sphere: regular inside, decaying outside

For **axial symmetry**, the potential is independent of the azimuthal angle. The natural angular functions are the Legendre polynomials:

$$P_0(\cos \theta) = 1, \quad P_1(\cos \theta) = \cos \theta, \quad P_2(\cos \theta) = \frac{1}{2}(3 \cos^2 \theta - 1).$$

Here θ is measured from the z axis.

Separation gives $(r^2 R')' = \ell(\ell + 1)R$. For $R = r^s$, this becomes $s(s + 1) = \ell(\ell + 1)$, so $s = \ell$ or $s = -\ell - 1$. The allowed functions are therefore

Mode	Angular pattern	Regular inside	Decays outside
$\ell = 0$	1	constant	$1/r$
$\ell = 1$	$\cos \theta$	$r \cos \theta$	$\cos \theta / r^2$
$\ell = 2$	$P_2(\cos \theta)$	$r^2 P_2(\cos \theta)$	$P_2(\cos \theta) / r^3$

Regularity applies when the interior contains no source at $r = 0$. For a localized source with zero potential at infinity, only the decaying exterior terms remain.

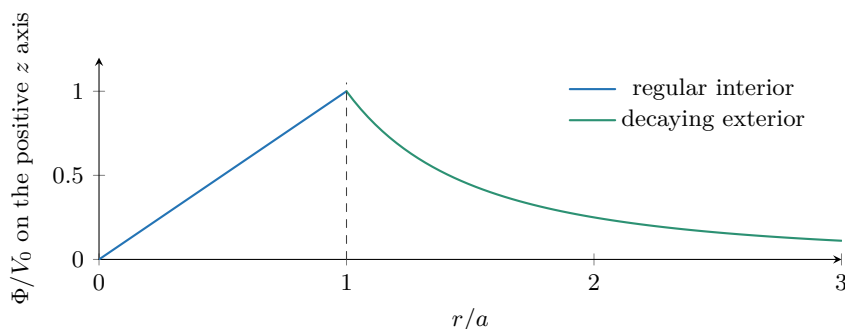
Example — the same boundary pattern, two solution regions

Prescribe $\Phi(a, \theta) = V_0 \cos \theta$ on a spherical boundary. This varying voltage requires separate electrodes, not one connected sphere.

Solution. The pattern is purely $\ell = 1$. In the two separate charge-free domains, with the indicated regularity or decay condition,

$$\Phi_{\text{in}} = V_0 \frac{r}{a} \cos \theta, \quad \Phi_{\text{out}} = V_0 \frac{a^2}{r^2} \cos \theta.$$

Inside, $r \cos \theta = z$, so $\mathbf{E}_{\text{in}} = -(V_0/a)\hat{\mathbf{z}}$. Outside, the potential has the dipole form with $p = 4\pi\epsilon_0 V_0 a^2$.



Both solutions match the same boundary value at $r = a$. Their radial slopes differ because they solve different domains.

Physical meaning

Week 3's metal sphere in a uniform applied field had induced potential $E_0 a^3 \cos \theta / r^2$. We can now identify its induced moment: $\mathbf{p}_{\text{ind}} = 4\pi\epsilon_0 a^3 \mathbf{E}_0$. Its net charge is zero, but its rearranged surface charge has a dipole moment.

OPTIONAL — The next surviving moment

Enrichment only. This material is not examined.

19 A quadrupole: when both Q and $\mathbf{p}$ vanish

The next term in the distance expansion is

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r^2} + \frac{3(\hat{\mathbf{r}} \cdot \mathbf{r}')^2 - r'^2}{2r^3} + \dots$$

It describes the **quadrupole** contribution. If $Q = 0$ and $\mathbf{p} = 0$, this is the next possible term; it too can vanish for a sufficiently symmetric source.

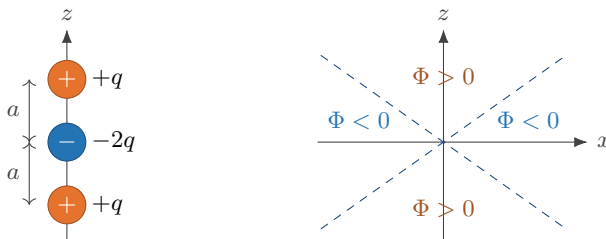
Example — a neutral symmetric three-charge source

Put $+q$ at $z = a$, $-2q$ at $z = 0$, and $+q$ at $z = -a$.

Solution. Both $Q = q - 2q + q = 0$ and $p_z = qa - qa = 0$. For each outer charge, $(\hat{\mathbf{r}} \cdot \mathbf{r}')^2 = a^2 \cos^2 \theta$. The two quadratic terms add:

$$\Phi_{\text{far}}(r, \theta) = k_e \frac{qa^2}{r^3} (3 \cos^2 \theta - 1), \quad r \gg a.$$

This potential is positive near the axis and negative near the equator when $q > 0$. Its generic field magnitude falls as $1/r^4$.



Source positions and the sign of its leading far potential. Dashed rays cut the nodal cones $3 \cos^2 \theta = 1$; they are not field lines.

Check / physical lesson

On the positive axis, $r > a$, the exact potential is

$$\Phi = k_e q \left(\frac{1}{r-a} + \frac{1}{r+a} - \frac{2}{r} \right) = \frac{2k_e qa^2}{r(r^2 - a^2)} \simeq \frac{2k_e qa^2}{r^3},$$

in agreement with the quadrupole result at $\theta = 0$.

Example — check the centered ring's first correction

For a uniformly charged ring of radius a , every element is distance $\sqrt{z^2 + a^2}$ from a point on the positive axis. Therefore

$$\Phi(z) = \frac{k_e Q}{\sqrt{z^2 + a^2}} = \frac{k_e Q}{z} \left[1 - \frac{a^2}{2z^2} + O\left(\frac{a^4}{z^4}\right) \right], \quad z \gg a.$$

There is no $1/z^2$ term, agreeing with $\mathbf{p} = 0$. The $-k_e Q a^2 / (2z^3)$ correction is quadrupolar.

OPTIONAL — A separation-of-variables dictionary

Enrichment only. This material is not examined.

20 Which functions belong to which geometry?

Laplace's equation has different natural building blocks in different coordinate systems. The geometry suggests the coordinates; the boundary conditions then select which of these building blocks survive.

geometry $\longrightarrow$ natural modes $\longrightarrow$ boundary conditions select the coefficients.

20.1 Cartesian coordinates

For rectangular or planar boundaries, separated solutions are built from

$$\sin(kx), \quad \cos(kx), \quad e^{\pm ky} \text{ or } \sinh(ky), \cosh(ky).$$

For example, in the half-space $y > 0$,

$$\Phi(x, 0) \propto \cos(kx), \quad \Phi \rightarrow 0 \quad (y \rightarrow \infty)$$

selects

$$\Phi(x, y) \propto e^{-ky} \cos(kx).$$

Thus

$$\begin{aligned} \text{zero at two walls} &\Rightarrow \sin(kx), \\ \text{periodic boundary pattern} &\Rightarrow \sin(kx), \cos(kx), \\ \text{decay away from a boundary} &\Rightarrow e^{-ky}. \end{aligned}$$

20.2 Cylindrical coordinates

For (ρ, ϕ, z) , periodicity around the axis gives the angular modes

$$\cos(m\phi), \quad \sin(m\phi), \quad m = 0, 1, 2, \dots$$

For a problem independent of z , the corresponding radial factors are

$$\rho^m, \quad \rho^{-m}, \quad m \geq 1,$$

while the axisymmetric mode $m = 0$ gives

$$1, \quad \ln \rho.$$

Hence a general z -independent harmonic potential has the form

$$\begin{aligned} \Phi(\rho, \phi) = & A_0 + B_0 \ln \rho \\ & + \sum_{m=1}^{\infty} (A_m \rho^m + B_m \rho^{-m}) (C_m \cos m\phi + D_m \sin m\phi). \end{aligned}$$

If the problem also varies along z , the radial functions become **Bessel functions**:

$$J_m(k\rho), \quad Y_m(k\rho), \quad I_m(k\rho), \quad K_m(k\rho).$$

Their useful properties are

$$\begin{aligned} J_m &: \text{regular at the axis,} \\ Y_m &: \text{singular at the axis,} \\ I_m &: \text{regular at the axis but grows outward,} \\ K_m &: \text{singular at the axis but decays outward.} \end{aligned}$$

So, as usual,

include the axis $\Rightarrow$ keep the regular branch, $\quad$ extend to infinity $\Rightarrow$ keep the decaying branch.

20.3 Spherical coordinates

For (r, θ, ϕ) , the angular building blocks are the **spherical harmonics**

$$Y_{\ell m}(\theta, \phi),$$

and the corresponding radial factors are

$$r^\ell, \quad r^{-(\ell+1)}.$$

Thus

$$\Phi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi).$$

For axial symmetry there is no ϕ dependence, and the angular functions reduce to the **Legendre polynomials**

$$P_\ell(\cos \theta) :$$

$$P_0 = 1, \quad P_1 = \cos \theta, \quad P_2 = \frac{1}{2}(3 \cos^2 \theta - 1), \dots$$

Therefore

$$\begin{aligned} 1 \Rightarrow \ell = 0 &\Rightarrow 1, 1/r, \\ \cos \theta \Rightarrow \ell = 1 &\Rightarrow r, 1/r^2, \\ P_2(\cos \theta) \Rightarrow \ell = 2 &\Rightarrow r^2, 1/r^3. \end{aligned}$$

Again,

$$r = 0 \text{ inside the domain} \Rightarrow r^\ell, \quad r \rightarrow \infty \text{ in the domain} \Rightarrow r^{-(\ell+1)}.$$

Check / physical lesson

A useful memory rule is

geometry	natural functions
Cartesian	$\sin, \cos, \exp, \sinh, \cosh$
Cylindrical	$\sin(m\phi), \cos(m\phi), J_m, Y_m, I_m, K_m$
Spherical	$Y_{\ell m}, P_\ell, r^\ell, r^{-(\ell+1)}$

Do not try to memorize every coefficient. First recognize the geometry and boundary pattern; then keep only the branches compatible with the domain.