

Electrodynamics & Relativity

Week 5: Steady Currents and Magnetic Fields

From moving charge to fields and forces

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Main idea. In Weeks 3 and 4, we found the electric field of stationary charge and used it to calculate energy and forces. We now let charge flow steadily: the current produces a magnetic field, and that field acts on moving charge. The working logic remains the same: **source, symmetry, field, force**.

1 **Steady current:** charge moves, the pattern stays fixed

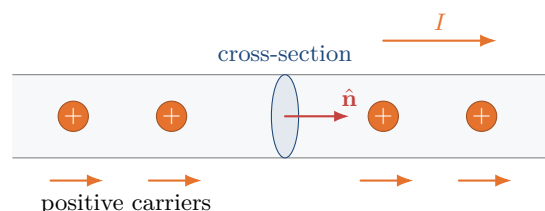
Magnetostatics describes **time-independent** currents and magnetic fields. The individual charge carriers move, but the current at each fixed point does not change with time.

1.1 **Current:** charge crossing a surface per second

Choose an oriented cross-section. If dQ_{cross} is the signed charge that crosses it during dt , the current is

$$I = \frac{dQ_{\text{cross}}}{dt}, \quad [I] = \text{A} = \text{C/s}. \quad (1.1)$$

This is charge *passing through* the section, not charge accumulating inside the wire. The current direction is the direction of positive-charge flow.



Current counts charge crossing the chosen section. A steady current can flow while the total charge in each piece of wire stays constant.

Example — what does 2 A mean?

A wire carries a constant current of 2.0 A. How much charge passes a cross-section in 3.0 s?

Solution. $Q_{\text{cross}} = I\Delta t = 6.0 \text{ C}$. This does *not* mean that 6.0 C is left behind in the wire.

Physical meaning

A current-carrying metal is not in the electrostatic equilibrium studied in Week 3. In an ordinary resistive wire, an electric field drives the steady current. A steady magnetic problem can therefore coexist with an electric field.

1.2 Current density: where does the current flow?

The vector $\mathbf{J}$ gives current per unit area perpendicular to the flow:

$$I = \int_S \mathbf{J} \cdot \hat{\mathbf{n}} \, dS, \quad [J] = \text{A/m}^2. \quad (1.2)$$

For uniform flow normal to a section of area A , $I = JA$. If carriers have number density n , signed charge q , and mean drift velocity $\mathbf{v}_d$, then

$$\mathbf{J} = nq\mathbf{v}_d.$$

For electrons, $q = -e$: [electron drift and conventional current point in opposite directions](#).

1.3 Charge conservation: a steady current cannot pile up

For any fixed volume V , the outward current equals the rate at which charge inside is lost:

$$\frac{d}{dt} \int_V \rho \, dV = - \oint_{\partial V} \mathbf{J} \cdot \hat{\mathbf{n}} \, dS.$$

The divergence theorem gives the **continuity equation**:

$$\left[\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \right] \xrightarrow{\text{steady}} \left[\nabla \cdot \mathbf{J} = 0 \right] \quad (1.3)$$

Thus the current entering any portion of a steady circuit equals the current leaving it. At a junction, the same statement gives $\sum I_{\text{in}} = \sum I_{\text{out}}$.

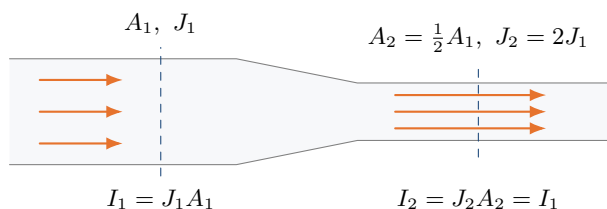
Example — a wire narrows

A steady current $I = 3.0 \text{ A}$ flows through a wire whose cross-section changes from $A_1 = 2.0 \text{ mm}^2$ to $A_2 = 1.0 \text{ mm}^2$. Assume the current density is uniform well away from the transition.

Solution. The same current crosses both sections:

$$J_1 = \frac{I}{A_1} = 1.5 \times 10^6 \text{ A/m}^2, \quad J_2 = \frac{I}{A_2} = 3.0 \times 10^6 \text{ A/m}^2.$$

If the carrier density is unchanged, the drift speed doubles as well.



[A smaller area means a larger current density, not a smaller steady current](#). Arrow lengths indicate the change in J .

Check / physical lesson

If 3 A entered a region but only 2 A left, its charge would grow at 1 C/s. That situation would not be steady.

2 Magnetic force: first find the direction

The force on a charge q moving with velocity $\mathbf{v}$ is the **Lorentz force**:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (2.1)$$

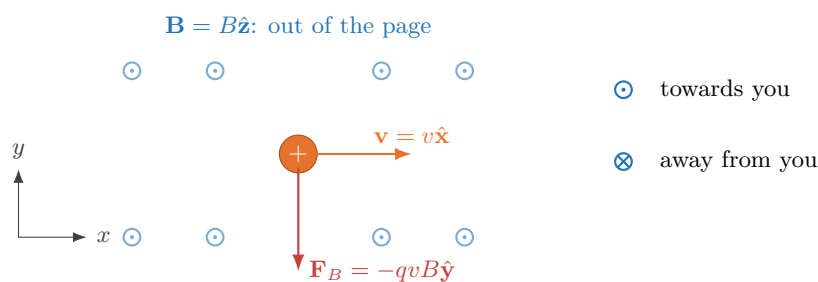
The magnetic part is

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}, \quad F_B = |q|vB \sin \theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{B}$. The unit of magnetic field is the tesla: $1 \text{ T} = 1 \text{ N}/(\text{A m})$.

Method / strategy

For a positive charge, use the right-hand rule for $\mathbf{v} \times \mathbf{B}$. For a negative charge, reverse that direction. The force is perpendicular to both vectors; it vanishes if the charge is at rest or moves parallel to the field.



A dot is an arrow tip coming out of the page; a cross is its tail going into the page. Here $\hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}$, so a positive charge moving right is pushed down.

Example — the same velocity, opposite charges

A proton and an electron each have instantaneous velocity $\mathbf{v} = (2.0 \times 10^5 \text{ m/s})\hat{\mathbf{x}}$ in $\mathbf{B} = (0.10 \text{ T})\hat{\mathbf{z}}$, with $\mathbf{E} = 0$.

Solution. With $e = 1.60 \times 10^{-19} \text{ C}$,

$$|\mathbf{F}_B| = evB = 3.2 \times 10^{-15} \text{ N}.$$

The proton is pushed along $-\hat{\mathbf{y}}$; the electron along $+\hat{\mathbf{y}}$. Their force magnitudes are equal, but their accelerations differ because their masses differ.

Physical meaning

An electric field acts on a stationary charge. A magnetic field acts on its motion: changing the velocity changes the magnetic force even at the same point in the same field.

2.1 A uniform field: bend the path without changing speed

The rate at which the magnetic force changes kinetic energy is

$$\frac{d}{dt} \left(\frac{1}{2} m v^2 \right) = \mathbf{F}_B \cdot \mathbf{v} = q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0.$$

A magnetic force alone does no work. It changes the direction of motion, not the speed. We use nonrelativistic motion here: $v \ll c$.

Example — circular motion

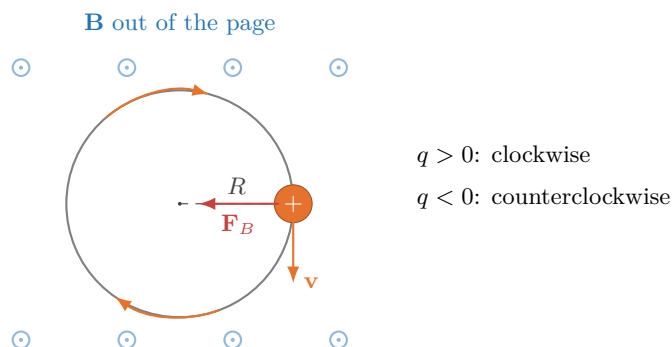
A charge enters a uniform field with $\mathbf{v} \perp \mathbf{B}$ and no other force. Find the orbit radius and the time for one turn.

Solution. The magnetic force supplies the centripetal force:

$$|q|vB = \frac{mv^2}{R} \implies \boxed{R = \frac{mv}{|q|B}}.$$

The angular speed and period follow from the geometry of the circle:

$$\boxed{\omega_c = \frac{v}{R} = \frac{|q|B}{m}}, \quad \boxed{T = \frac{2\pi}{\omega_c} = \frac{2\pi m}{|q|B}}.$$



At the rightmost point, a positive charge moving downward feels an inward force: $(-\hat{y}) \times \hat{z} = -\hat{x}$. Reversing the charge reverses the sense of rotation.

For the proton on the previous page, using $m_p = 1.67 \times 10^{-27}$ kg gives $R = 2.09$ cm and $T = 0.656$ μ s.

What if the velocity also has a component along the field?

Write $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$. Only $\mathbf{v}_\perp$ enters the cross product. The transverse motion is a circle of radius $mv_\perp/(|q|B)$, while $\mathbf{v}_\parallel$ stays constant. Together they form a helix, advancing a distance $|v_\parallel|T$ along the field axis per turn.

Check / physical lesson

Doubling $v_\perp$ doubles the radius but leaves the period unchanged. Doubling B halves both the radius and the period.

2.2 Force on a wire: add the forces on its carriers

A current is moving charge, so a wire in an applied magnetic field also feels a force. For a short element directed along the conventional current,

$$\boxed{d\mathbf{F} = I d\boldsymbol{\ell} \times \mathbf{B}_{\text{ext}}.} \quad (2.2)$$

Here $\mathbf{B}_{\text{ext}}$ is the field produced by **other sources**. It is not the field of this same wire element acting on itself.

From one carrier to a wire element

Take positive carriers with density n , charge q , and drift speed v_d in a wire of cross-section A . A length $d\ell$ contains $nA d\ell$ carriers, so its force is

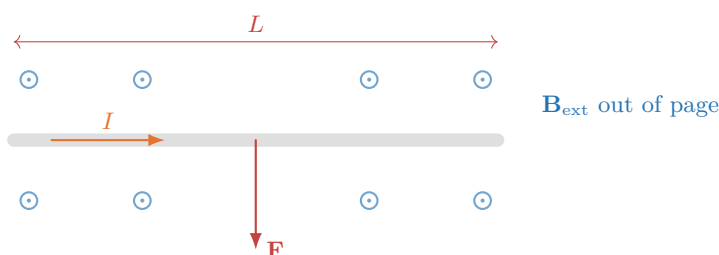
$$d\mathbf{F} = (nA d\ell)q\mathbf{v}_d \times \mathbf{B}_{\text{ext}} = \underbrace{nqAv_d}_I d\boldsymbol{\ell} \times \mathbf{B}_{\text{ext}}.$$

For negative carriers both q and the drift direction reverse relative to conventional current, giving the same current formula.

For a straight wire of directed length $\mathbf{L}$ in a uniform field,

$$\boxed{\mathbf{F} = I\mathbf{L} \times \mathbf{B}_{\text{ext}}.}$$

For a curved wire or a nonuniform field, keep the integral $\mathbf{F} = I \int d\boldsymbol{\ell} \times \mathbf{B}_{\text{ext}}$.



Current to the right and field out of the page give force downward. **Use the current direction in $\mathbf{L} \times \mathbf{B}_{\text{ext}}$, even when electrons carry the current.**

Example — a straight segment in a uniform field

A horizontal segment of length $L = 0.10$ m carries $I = 3.0$ A to the right. A field $B = 0.20$ T points out of the page.

Solution. The vectors are perpendicular, so

$$F = ILB = 0.060 \text{ N}, \quad \text{directed downward.}$$

Rotating the segment until it is parallel to the field reduces the force to zero.

The same statement for a volume current

For a continuous distribution, the magnetic force per unit volume is $\mathbf{f}_B = \mathbf{J} \times \mathbf{B}_{\text{ext}}$. Integrating $\mathbf{f}_B dV$ adds the forces throughout the material.

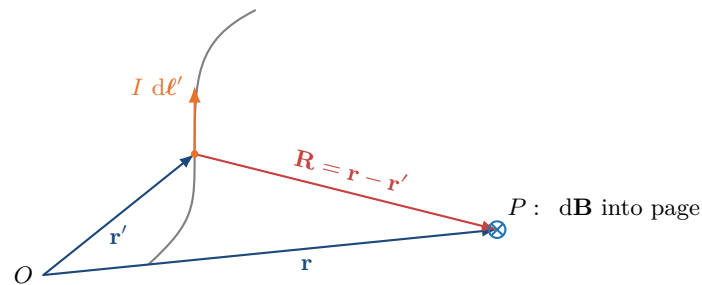
3 Biot–Savart: build the field from the current

We now reverse the question: given the steady current, what magnetic field does it produce? For a thin wire in vacuum,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\text{wire}} \frac{d\ell' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (3.1)$$

Here $\mu_0 \simeq 4\pi \times 10^{-7} \text{ N/A}^2$ is the vacuum permeability. As in the Coulomb integral, primed coordinates label the **source**; the unprimed point is where we observe the field.

$\mathbf{r}'$	Position of a current element.
$\mathbf{r}$	Fixed observation point while we integrate.
$\mathbf{R} = \mathbf{r} - \mathbf{r}'$	Vector from the source to the observer .
$d\ell'$	Wire element directed along conventional current.



The element points up and the observer lies to its right, so $d\ell' \times \mathbf{R}$ points into the page. **The field contribution is perpendicular to the plane containing the current element and the observer.**

Method / strategy

1. Sketch the source and observer; define $\mathbf{R}$. **2.** Find the cross-product direction before integrating. **3.** Use symmetry to cancel components. **4.** Integrate the surviving component and check a simple limit.

Example — one element, two observation directions

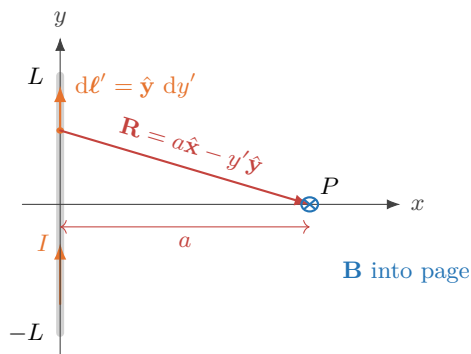
A small element points along $+\hat{\mathbf{y}}$. At an observer directly above it, $\mathbf{R} \parallel d\ell'$ and its contribution is zero. At an observer directly to its right, $\mathbf{R} = R\hat{\mathbf{x}}$, so

$$d\mathbf{B} = -\frac{\mu_0 I d\ell'}{4\pi R^2} \hat{\mathbf{z}}.$$

This is a contribution from one element; the total field adds all elements.

3.1 A finite straight segment: keep the geometry explicit

Consider the segment $-L \leq y' \leq L$ on the y axis, with current in $+\hat{\mathbf{y}}$. Find its contribution at $P = (a, 0, 0)$, where $a > 0$.



Every element of this segment contributes into the page at P . The perpendicular distance is a , but the element-to-observer distance is $\sqrt{a^2 + y'^2}$. The rest of the circuit is not shown.

The cross product fixes the sign

With $d\ell' = \hat{\mathbf{y}} dy'$,

$$d\ell' \times \mathbf{R} = \hat{\mathbf{y}} \times (a\hat{\mathbf{x}} - y'\hat{\mathbf{y}}) dy' = -a\hat{\mathbf{z}} dy'.$$

Therefore

$$\mathbf{B}(P) = -\hat{\mathbf{z}} \frac{\mu_0 I a}{4\pi} \int_{-L}^L \frac{dy'}{(a^2 + y'^2)^{3/2}},$$

$$\int \frac{dy'}{(a^2 + y'^2)^{3/2}} = \frac{y'}{a^2 \sqrt{a^2 + y'^2}}.$$

Inserting the endpoints gives

$$\mathbf{B}(P) = -\hat{\mathbf{z}} \frac{\mu_0 I}{2\pi a} \frac{L}{\sqrt{a^2 + L^2}}. \quad (3.2)$$

Example — how long is “long”?

If $L = a$, the field magnitude is $1/\sqrt{2}$ of the infinite-wire value. If $L = 10a$, the factor is $10/\sqrt{101} = 0.995$.

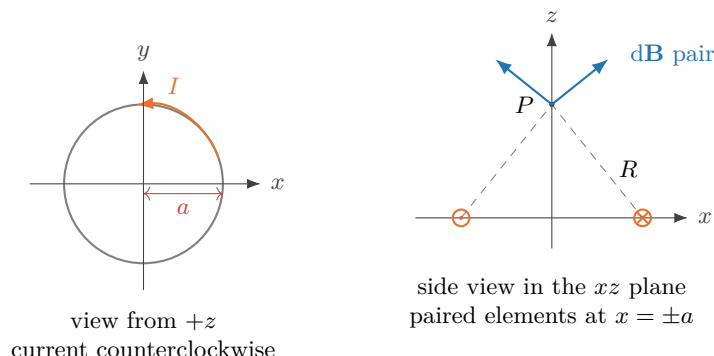
Check. As $L \rightarrow \infty$, $B \rightarrow \mu_0 I / (2\pi a)$; as $L \rightarrow 0$, this segment’s contribution vanishes.

Physical meaning

A steady current cannot end at the endpoints of an isolated segment. Equation (3.2) gives the contribution of the straight part of a complete circuit; other parts may contribute additional field.

3.2 A circular loop: add the axial components

A loop of radius a lies in the xy plane. Its current is counterclockwise when viewed from $+z$. Find the field at $P = (0, 0, z)$. Each element is a distance $R = \sqrt{a^2 + z^2}$ from P .



At paired elements, the transverse field components cancel and the axial components add. In the side view, the right-hand current points into the page and the left-hand current points out.

Why the axial projection supplies a second factor of a

The tangent $d\ell'$ is perpendicular to $\mathbf{R}$, so $dB = \mu_0 I d\ell' / (4\pi R^2)$. At the right-hand element, $\mathbf{R} = -a\hat{\mathbf{x}} + z\hat{\mathbf{z}}$ and $d\ell' = \hat{\mathbf{y}} d\ell'$, giving

$$d\ell' \times \mathbf{R} = (z\hat{\mathbf{x}} + a\hat{\mathbf{z}}) d\ell'.$$

The axial component is therefore a fraction a/R of the magnitude: $dB_z = (a/R) dB$. Since a and R are constant around the loop,

$$B_z = \frac{\mu_0 I a}{4\pi R^3} \oint d\ell' = \frac{\mu_0 I a}{4\pi R^3} (2\pi a).$$

$$\mathbf{B}(0, 0, z) = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \hat{\mathbf{z}}. \quad (3.3)$$

Example — at the centre and one radius away

For $a = 0.10$ m and $I = 5.0$ A,

$$B(0) = \frac{\mu_0 I}{2a} = 31.4 \mu\text{T}, \quad B(a) = \frac{B(0)}{2^{3/2}} = 11.1 \mu\text{T}.$$

Both fields point along $+\hat{\mathbf{z}}$. On the negative z axis the axial field also points along $+\hat{\mathbf{z}}$; reversing the current reverses the field everywhere.

Check / physical lesson

Far from the loop, $|z| \gg a$, the field falls as $B \simeq \mu_0 I a^2 / (2|z|^3)$. This $1/r^3$ behaviour is the magnetic counterpart of the dipole field in Week 4.

4 Flux and circulation: ask two different questions

4.1 Magnetic flux: how much field crosses a surface?

For an oriented surface S , the magnetic flux is

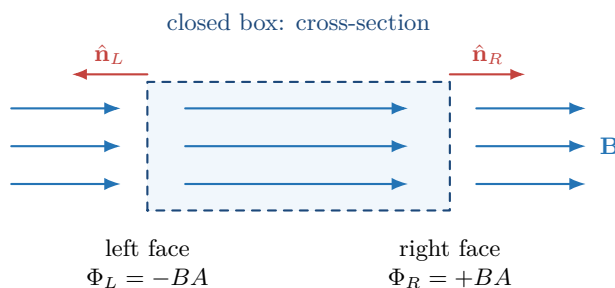
$$\Phi_B = \int_S \mathbf{B} \cdot \hat{\mathbf{n}} \, dS.$$

For a flat surface of area A in a uniform field, $\Phi_B = BA \cos \theta$, where θ is the angle between the field and the **surface normal**. Flux has units of weber: $1 \text{ Wb} = 1 \text{ T m}^2$.

For a **closed** surface, with the normal pointing outward, the magnetic flux is always zero:

$$\oint_{\partial V} \mathbf{B} \cdot \hat{\mathbf{n}} \, dS = 0 \iff \nabla \cdot \mathbf{B} = 0. \quad (4.1)$$

The equivalence follows from the divergence theorem, since the closed-surface statement holds for every volume. There are no magnetic charge sources in classical electromagnetism: magnetic field lines have no beginnings or ends.



A uniform field passes through the box. It enters one face and leaves the other; on the remaining faces it is tangential.
Zero net flux does not mean zero field.

Example — a nonzero field with zero closed-surface flux

The field is $B = 2.0 \text{ mT}$ and each end face of the box has area $A = 10 \text{ cm}^2 = 10^{-3} \text{ m}^2$.

Solution. The two end-face fluxes are $-2.0 \mu\text{Wb}$ and $+2.0 \mu\text{Wb}$. The other faces contribute zero, so the total is zero. The flux through either end face alone is not zero.

Physical meaning

Compare the source equations:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \quad \nabla \cdot \mathbf{B} = 0.$$

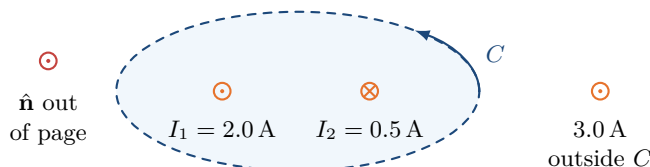
Electric charge can produce net outward electric flux. Current produces magnetic circulation, which we examine next.

4.2 Ampère's law: how much field runs around a loop?

For steady currents in vacuum,

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}, \quad I_{\text{enc}} = \int_S \mathbf{J} \cdot \hat{\mathbf{n}} dS. \quad (4.2)$$

The surface S spans the closed curve C . Curl the fingers of your right hand along the chosen traversal of C ; your thumb gives $\hat{\mathbf{n}}$. Currents through S are then counted with their signs.



Counterclockwise traversal chooses the outward normal shown. **Count only current piercing the spanning surface, with out-of-page current positive and into-page current negative.**

Example — circulation is fixed, even when the field is not simple

For the loop above, $I_{\text{enc}} = 2.0 - 0.5 = 1.5 \text{ A}$, so

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = 1.5\mu_0 \text{ A} \simeq 1.88 \times 10^{-6} \text{ T m}.$$

The outside current can change $\mathbf{B}$ at points on C , but contributes zero net circulation around this loop. We cannot deduce a single value of B from the integral without additional symmetry.

The local statement: apply Stokes' theorem

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{B}) \cdot \hat{\mathbf{n}} dS = \mu_0 \int_S \mathbf{J} \cdot \hat{\mathbf{n}} dS.$$

Since this holds for every suitable surface, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Compare electrostatics, where $\nabla \times \mathbf{E} = 0$ and $\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0$.

Method / strategy

When does Ampère's law calculate the field easily? Choose a loop on which symmetry makes the tangential field constant, or makes $\mathbf{B} \cdot d\boldsymbol{\ell} = 0$. Otherwise use Biot–Savart. An Amperian loop is an imaginary integration path, not a physical wire.

	Flux	Circulation
Integral	$\int_S \mathbf{B} \cdot \hat{\mathbf{n}} dS$	$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell}$
Component	Normal to a surface	Tangent to a curve

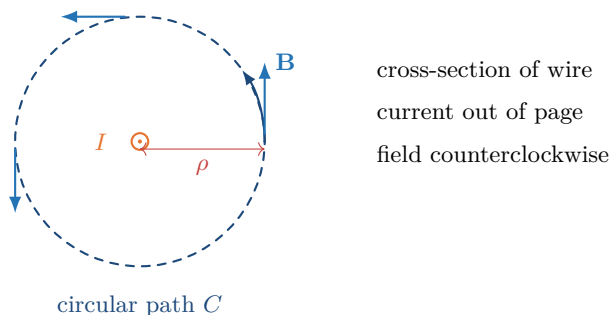
5 Use symmetry: five applications of Ampère's law

5.1 An infinite straight wire: a circular integration path

A thin wire carries current I along $+\hat{\mathbf{z}}$. The field circulates around the wire by the right-hand rule. Rotational and translational symmetry make its magnitude depend only on the distance ρ from the axis:

$$\mathbf{B} = B(\rho)\hat{\phi}.$$

Choose a circle of radius ρ centred on the wire. On that circle, $\mathbf{B}$ is tangential and has constant magnitude.



The circular path follows the field, so $\mathbf{B} \cdot d\boldsymbol{\ell} = B d\ell$ at every point. This is why the unknown $B(\rho)$ can leave the integral.

The integral becomes field times circumference

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = B(\rho) \oint_C d\ell = B(\rho) 2\pi\rho = \mu_0 I.$$

$$\mathbf{B}(\rho) = \frac{\mu_0 I}{2\pi\rho} \hat{\phi}. \quad (5.1)$$

The magnitude falls as $1/\rho$: the fixed circulation is spread around a larger circumference. This agrees with the long-segment limit of Biot–Savart on page 7.

Example — field near a straight wire

For $I = 5.0$ A at $\rho = 2.0$ cm,

$$B = \frac{(4\pi \times 10^{-7})(5.0)}{2\pi(0.020)} = 5.0 \times 10^{-5} \text{ T} = 50 \mu\text{T}.$$

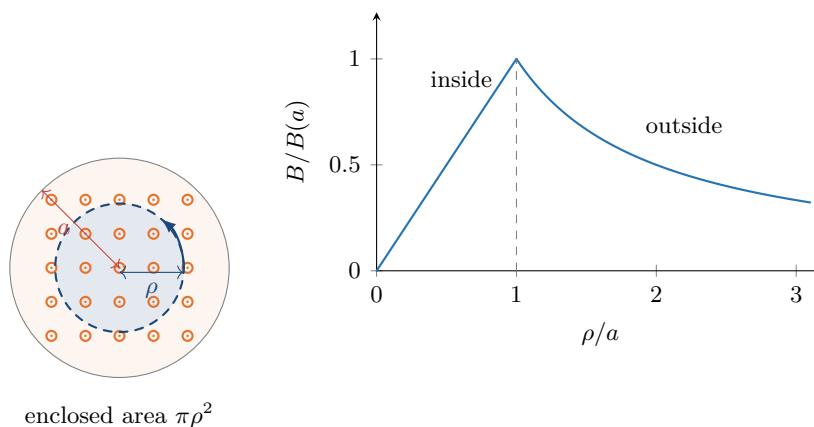
At twice the distance the field is $25 \mu\text{T}$. The direction still follows the circular field lines.

Check / physical lesson

The formula applies outside a wire whose radius is negligible compared with ρ . Its apparent divergence as $\rho \rightarrow 0$ tells us that a zero-radius wire is an idealisation. A wire with finite radius resolves it.

5.2 A solid cylindrical wire: count only the enclosed current

A long wire of radius a carries current I uniformly over its cross-section: $\mathbf{J} = I\hat{\mathbf{z}}/(\pi a^2)$. Cylindrical symmetry again gives circular field lines. The change is in I_{enc} , not in the choice of Amperian path.



The shaded disk encloses only part of the current. Inside, enclosed current grows as ρ^2 while circumference grows as ρ ; therefore B grows linearly.

Use the enclosed area before applying Ampère's law

For $\rho < a$, $I_{\text{enc}} = J\pi\rho^2 = I\rho^2/a^2$. For $\rho \geq a$, the full current is enclosed: $I_{\text{enc}} = I$. In both regions, $B(\rho)2\pi\rho = \mu_0 I_{\text{enc}}$, so

$$\mathbf{B}(\rho) = \begin{cases} \frac{\mu_0 I \rho}{2\pi a^2} \hat{\phi}, & 0 \leq \rho < a, \\ \frac{\mu_0 I}{2\pi \rho} \hat{\phi}, & \rho \geq a. \end{cases} \quad (5.2)$$

At the axis, $\mathbf{B} = \mathbf{0}$ by the continuous limit; the azimuthal direction need not be defined there.

Example — two points with the same field magnitude

Take $I = 5.0$ A and $a = 2.0$ mm. The surface field is $B(a) = 0.50$ mT. At $\rho = a/2$, $B = 0.25$ mT; at $\rho = 2a$, $B = 0.25$ mT too. One point encloses less current, while the other lies on a longer path.

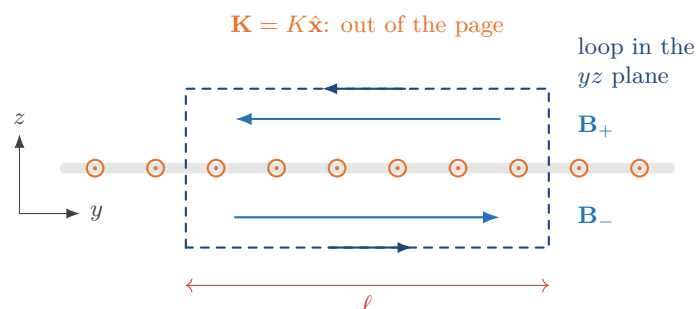
Check / physical lesson

The two formulas agree at $\rho = a$: the field is continuous. It is not zero inside a current-carrying conductor; the electrostatic condition $\mathbf{E} = 0$ inside a metal at equilibrium does not impose $\mathbf{B} = 0$.

5.3 An infinite current sheet: two sides, opposite fields

A uniform sheet lies at $z = 0$ and carries surface current $\mathbf{K} = K\hat{\mathbf{x}}$. Surface current density has units A/m: a strip of width ℓ measured perpendicular to the flow carries current $K\ell$.

Translations along the sheet make the field independent of x and y . The right-hand rule and reflection symmetry give equal, opposite fields on the two sides: along $-\hat{\mathbf{y}}$ above and $+\hat{\mathbf{y}}$ below. We calculate the sheet's own field, with no added external field.



The loop lies along the field on its long sides, and the current pierces its surface. Counterclockwise traversal chooses $+\hat{\mathbf{x}}$ as the surface normal; both long sides contribute positively.

Why the two long sides add

The short sides have $\mathbf{B} \perp d\ell$, so contribute zero. On each long side the field follows the traversal, giving

$$\oint_C \mathbf{B} \cdot d\ell = B\ell + B\ell = 2B\ell, \quad I_{\text{enc}} = K\ell.$$

Ampère's law therefore gives $2B\ell = \mu_0 K\ell$.

$$\mathbf{B}(z) = \begin{cases} -\frac{\mu_0 K}{2} \hat{\mathbf{y}}, & z > 0, \\ +\frac{\mu_0 K}{2} \hat{\mathbf{y}}, & z < 0. \end{cases} \quad (5.3)$$

Example — current per width

A uniform sheet carries 2.0 A through every 0.10 m of transverse width. Then $K = 20 \text{ A/m}$ and

$$B = \frac{\mu_0 K}{2} = 12.6 \mu\text{T}$$

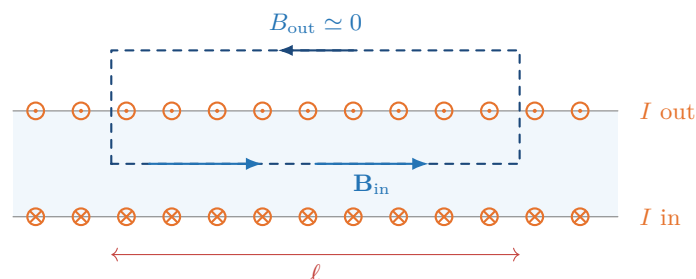
on each side, with the opposite directions shown above.

Check / physical lesson

The ideal sheet's field is independent of distance from the sheet. The rectangular loop may be made taller without changing either the enclosed current or the two nonzero contributions to the circulation.

5.4 A long solenoid: count the turns pierced by the loop

A solenoid has n turns per unit length, each carrying current I . We idealise the closely spaced windings as circular turns on a long cylinder. Away from the ends, the field is nearly uniform and parallel to the axis inside; the exterior field is small.



Cross-section through the axis. The loop's bottom side lies inside and follows the field; its top side lies outside. The spanning surface cuts $n\ell$ turns, so it encloses $n\ell I$, not just I .

Compare the interior and exterior fields

Choose the positive axial direction to the right. The transverse sides of the rectangle contribute zero, while the axial sides give

$$(B_{\text{in}} - B_{\text{out}})\ell = \mu_0(n\ell)I, \quad B_{\text{in}} - B_{\text{out}} = \mu_0 nI.$$

In the infinite-solenoid model, symmetry and the current-free exterior make the exterior axial field spatially constant. With no applied field, it must vanish far away, hence $B_{\text{out}} = 0$.

For a long, finite solenoid away from its ends,

$$B_{\text{in}} \simeq \mu_0 nI, \quad B_{\text{out}} \simeq 0. \quad (5.4)$$

The right-hand grip rule fixes the axial direction: curl your fingers with the winding current; your thumb points along the interior field.

Example — turn count versus turn density

A coil has $N = 500$ turns over length $L = 0.50$ m and carries $I = 2.0$ A. Its radius is much smaller than L .

Solution. The turn density is $n = N/L = 1000 \text{ m}^{-1}$, so

$$B_{\text{in}} \simeq \mu_0 nI = 2.51 \text{ mT}.$$

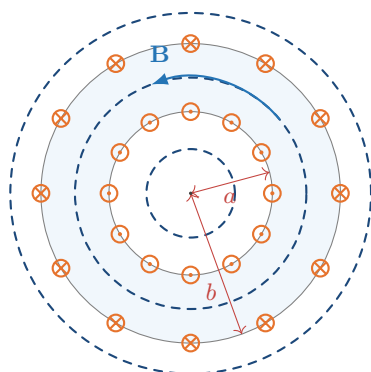
A rectangular loop of axial length $\ell = 0.10$ m encloses $n\ell = 100$ turns and thus $I_{\text{enc}} = 200$ A. This is a sum over crossings; the current in each wire remains 2.0 A.

Physical meaning

The small exterior field is a property of the long-solenoid geometry and the boundary condition, not a consequence of $I_{\text{enc}} = 0$ alone. A finite solenoid has a return field outside, especially near its ends.

5.5 An ideal toroid: a solenoid bent into a ring

An ideal toroid has N uniformly distributed turns carrying current I . Take a rectangular cross-section with inner radius a and outer radius b . In the central plane, let ρ be distance from the *symmetry axis* of the whole ring. The magnetic field is azimuthal.



$\rho < a$: no crossings

$$I_{\text{enc}} = 0$$

$a < \rho < b$: N outward crossings

$$I_{\text{enc}} = NI$$

$\rho > b$: outward and return crossings

$$I_{\text{enc}} = NI - NI = 0$$

A cut through the central plane. Dots and crosses are the two sides of each winding, piercing this plane in opposite directions. A circular path within the annulus encloses one side of each turn; an exterior path encloses both.

Ampère's law gives $B(\rho) 2\pi\rho = \mu_0 I_{\text{enc}}$. With ideal continuous windings and no applied external field,

$$\mathbf{B}(\rho) = \begin{cases} \mathbf{0}, & \rho < a, \\ \frac{\mu_0 NI}{2\pi\rho} \hat{\phi}, & a < \rho < b, \\ \mathbf{0}, & \rho > b. \end{cases} \quad (5.5)$$

These regions refer to the central plane shown; the nonzero field occupies the wound toroidal volume. Real, discrete windings produce some leakage field.

Example — is the field uniform across the ring?

For $N = 200$, $I = 1.0$ A, $a = 4.0$ cm, and $b = 6.0$ cm, the field just inside the inner and outer edges is

$$B(a^+) = 1.00 \text{ mT}, \quad B(b^-) = 0.667 \text{ mT}.$$

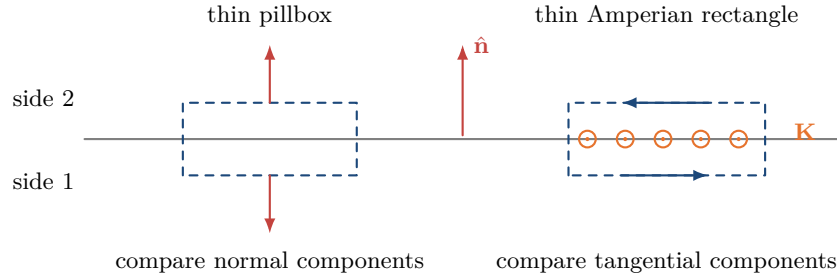
It is stronger at the inner edge because the circular path is shorter.

Check / physical lesson

For a narrow ring of mean radius R , $\rho \simeq R$ throughout the winding. Then $B \simeq \mu_0 NI / (2\pi R) = \mu_0 nI$ with $n = N / (2\pi R)$: the long-solenoid result reappears.

6 At an interface: which components can jump?

Consider vacuum on both sides of a surface carrying a possible sheet current $\mathbf{K}$. Let $\hat{\mathbf{n}}$ point from side 1 to side 2, and write $\Delta\mathbf{B} = \mathbf{B}_2 - \mathbf{B}_1$. Use the same shrinking-volume and shrinking-loop logic as for electric boundary conditions in Week 3.



Let the height of each construction tend to zero. **Flux compares the two faces of the pillbox; circulation compares the two long sides of the rectangle.** The example sheet current on the right points out of the page.

6.1 Normal component: continuous

The side-wall flux vanishes as the pillbox height shrinks. Its two faces give

$$(\mathbf{B}_2 \cdot \hat{\mathbf{n}})A - (\mathbf{B}_1 \cdot \hat{\mathbf{n}})A = 0,$$

so

$$\boxed{\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0.} \quad (6.1)$$

6.2 Tangential component: a sheet current creates a jump

As the rectangle shrinks, the short sides contribute zero, but a sheet current still pierces its surface. If $\mathbf{t}$ is a unit tangent along the lower side, the loop normal is $\mathbf{t} \times \hat{\mathbf{n}}$. Ampère's law gives

$$-(\Delta\mathbf{B} \cdot \mathbf{t})\ell = \mu_0 [\mathbf{K} \cdot (\mathbf{t} \times \hat{\mathbf{n}})]\ell.$$

Using $\mathbf{K} \cdot (\mathbf{t} \times \hat{\mathbf{n}}) = -\mathbf{t} \cdot (\mathbf{K} \times \hat{\mathbf{n}})$, and allowing any tangential direction $\mathbf{t}$, we obtain

$$\boxed{\hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0 \mathbf{K}}, \quad \boxed{\Delta\mathbf{B}_{\parallel} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}.} \quad (6.2)$$

The cross product fixes the direction of the jump; it is not generally parallel to the sheet current.

Example — read the current from a field jump

Let $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and suppose $\Delta\mathbf{B} = -\beta\hat{\mathbf{y}}$, with $\beta > 0$. Then

$$\mathbf{K} = \frac{1}{\mu_0} \hat{\mathbf{z}} \times (-\beta\hat{\mathbf{y}}) = \frac{\beta}{\mu_0} \hat{\mathbf{x}}.$$

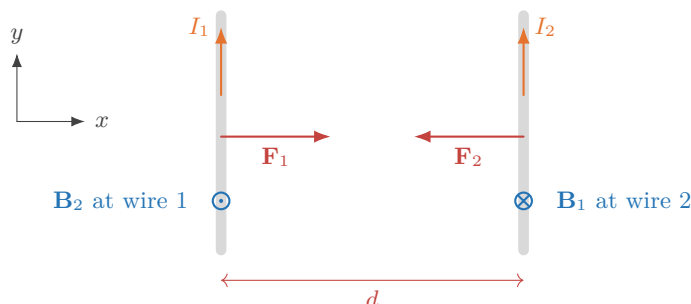
The required current points along $+x$, consistent with the sheet on page 13.

Check / physical lesson

If there is no sheet current, both components of $\mathbf{B}$ are continuous in vacuum. A finite volume-current density may change abruptly without creating a jump in $\mathbf{B}$, as at the surface of the solid wire.

7 Two parallel currents: put field and force together

Two long straight wires are separated by distance d . Their currents I_1 and I_2 both point upward. First find the field of wire 1 at wire 2; then use that field in the force law for wire 2.



At wire 2 the field of wire 1 points into the page. Upward current crossed with inward field gives a force to the left: the wires attract. Each force uses the *other* wire's field.

Field first, then force

At wire 2,

$$\mathbf{B}_1 = -\frac{\mu_0 I_1}{2\pi d} \hat{\mathbf{z}}.$$

For a length L of wire 2,

$$\mathbf{F}_2 = I_2 L \hat{\mathbf{y}} \times \mathbf{B}_1 = -\frac{\mu_0 I_1 I_2 L}{2\pi d} \hat{\mathbf{x}}.$$

The force on wire 1 has the same magnitude and the opposite direction.

$$\boxed{\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi d}} \quad (I_1, I_2 \text{ are current magnitudes}). \quad (7.1)$$

Parallel currents attract; antiparallel currents repel. Reversing one current reverses its force and also reverses the field it produces at the other wire.

Example — force per metre

For $I_1 = I_2 = 5.0$ A and $d = 2.0$ cm,

$$\frac{F}{L} = \frac{(4\pi \times 10^{-7})(5.0)(5.0)}{2\pi(0.020)} = 2.5 \times 10^{-4} \text{ N/m}.$$

A 0.50 m length feels 1.25×10^{-4} N of attraction, neglecting end effects.

Method / strategy

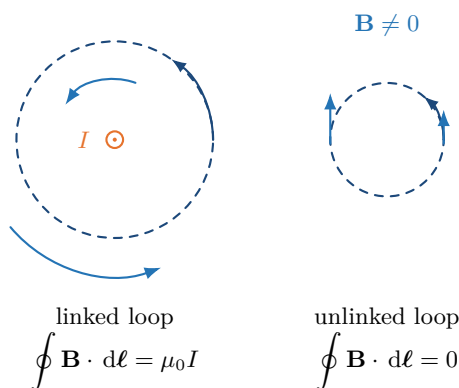
The same sequence works in a new geometry: **identify the source current**, find its field using symmetry and Ampère or a Biot–Savart integral, then apply $q\mathbf{v} \times \mathbf{B}$ or $I \, d\boldsymbol{\ell} \times \mathbf{B}$ to the object that feels the force.

OPTIONAL — Local curl and a loop that links a current

Further detail, following the main development.

8 No current here does not mean no circulation around there

Outside an infinitely thin straight wire, $\mathbf{J} = 0$, and hence $\nabla \times \mathbf{B} = 0$. Yet a loop surrounding the wire has circulation $\mu_0 I$. There is no contradiction: the two statements refer to **different regions**.



Both curves lie in current-free space. Only the left one links the wire. **On the right, a nonzero field has zero total circulation; its contributions around the curve cancel.**

8.1 Where does Stokes' theorem look?

Stokes' theorem relates the line integral on a boundary to the curl on a *whole spanning surface*:

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{B}) \cdot \hat{\mathbf{n}} \, dS.$$

Every surface spanning the linked loop intersects the wire. It is therefore incorrect to set the curl to zero on the entire surface. For the loop on the right, a spanning disk avoids the current and gives zero circulation.

Example — the danger of dividing by a path length

For the unlinked loop, Ampère's law gives $\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = 0$. It does not give $B \times \text{circumference} = 0$: the field is neither constant nor everywhere tangent to this displaced circle. The actual field is still $\mu_0 I / (2\pi\rho)$ at every point of the loop.

Why one magnetic scalar potential cannot cover the whole exterior

In a small current-free neighbourhood, one may write $\mathbf{B} = -\nabla\Psi$. Around the wire this would require

$$\Psi = -\frac{\mu_0 I}{2\pi} \phi + \text{constant}.$$

After one circuit, $\phi \rightarrow \phi + 2\pi$, the value changes by $-\mu_0 I$ even though the position is the same. Thus this scalar potential cannot be single-valued throughout the whole region outside the wire. The obstruction is the loop linking the excluded current.

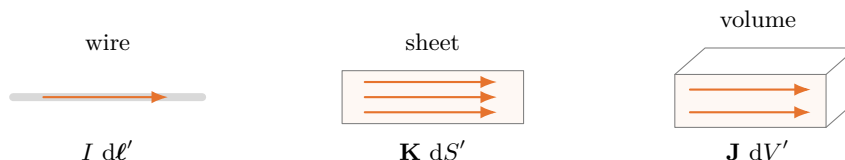
OPTIONAL — Extended currents and the divergence of the field
 Further detail, following the main development.

9 The same source integral for a wire, sheet, or volume

The thin-wire description is an idealisation. For a current spread over a surface or throughout a volume, replace the current element in Biot–Savart by the corresponding surface or volume element:

Source	Current element in the integral	Density units
Thin wire	$I \, d\ell'$	I : A
Thin sheet	$\mathbf{K}(\mathbf{r}') \, dS'$	K : A/m
Volume	$\mathbf{J}(\mathbf{r}') \, dV'$	J : A/m ²

All three current elements have units of A m. The geometric kernel is unchanged: take the cross product with $\mathbf{R}/R^3$, where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.



Different descriptions of how current occupies space. [Choose one model appropriate to the source; do not insert \$I\$, \$K\$, and \$J\$ as interchangeable quantities.](#)

For example, a thin layer of thickness t with uniform tangential volume current $\mathbf{J}$ has surface current $\mathbf{K} = t\mathbf{J}$. Integrating through the thickness converts $\mathbf{J} \, dV'$ into $\mathbf{K} \, dS'$.

9.1 Write the volume-current field as a curl

For a localized steady source,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV'.$$

Set $G = 1/|\mathbf{r} - \mathbf{r}'|$. The gradient with respect to the observation point is $\nabla G = -\mathbf{R}/R^3$. Because $\mathbf{J}(\mathbf{r}')$ is independent of $\mathbf{r}$,

$$\nabla \times (G\mathbf{J}') = \nabla G \times \mathbf{J}' = \frac{\mathbf{J}' \times \mathbf{R}}{R^3}.$$

Thus, defining the auxiliary vector field

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV'}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (9.1)$$

immediately gives $\boxed{\nabla \cdot \mathbf{B} = 0}$, since the divergence of a curl is zero. This $\mathbf{A}$ is the **magnetic vector potential**. Different vector potentials can give the same $\mathbf{B}$: this is **gauge freedom**. Week 6 develops this freedom and the magnetic dipole description of a small current loop.

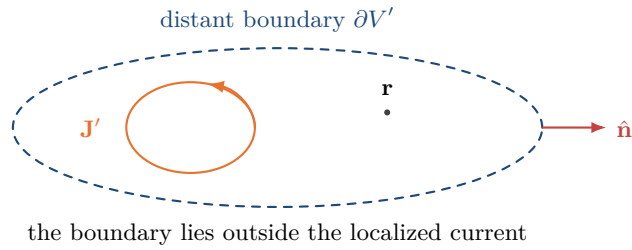
OPTIONAL — Why the curl equation follows from Biot–Savart
Further detail, following the main development.

10 Use charge conservation once more

The remaining step is to show that the volume-current construction gives $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Keep track of the two derivatives: ∇ acts on the observation point $\mathbf{r}$; ∇' acts on the source point $\mathbf{r}'$.

10.1 The source is steady and localized

The steady-source condition is $\nabla' \cdot \mathbf{J}' = 0$. Localisation means that the current vanishes, or decays sufficiently fast, on a distant boundary surrounding the source.



The source integral surrounds the whole current distribution. [Charge conservation removes a volume term](#); [localisation removes a boundary term](#). Both assumptions are used below.

First show that the constructed vector potential has zero divergence

Using $\nabla G = -\nabla' G$ and integration by parts,

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \frac{\mu_0}{4\pi} \int \mathbf{J}' \cdot \nabla G \, dV' \\ &= -\frac{\mu_0}{4\pi} \oint_{\partial V'} G \mathbf{J}' \cdot \hat{\mathbf{n}}' \, dS' + \frac{\mu_0}{4\pi} \int G (\nabla' \cdot \mathbf{J}') \, dV' = 0. \end{aligned}$$

The surface term vanishes at a distant boundary; the volume term vanishes for a steady current. This establishes $\nabla \cdot \mathbf{A} = 0$ for the particular $\mathbf{A}$ in Eq. (9.1).

10.2 The Coulomb kernel supplies the local source

Use the same Green-function identity that underlies Poisson's equation:

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta^{(3)}(\mathbf{r} - \mathbf{r}').$$

Applying it component by component gives

$$\nabla^2 \mathbf{A} = \frac{\mu_0}{4\pi} \int \mathbf{J}' \nabla^2 G \, dV' = -\mu_0 \mathbf{J}(\mathbf{r}).$$

Finally, the curl-of-curl identity yields

$$\boxed{\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.} \quad (10.1)$$