

Electrodynamics & Relativity

Week 6: Magnetic Vector Potential and Dipoles

Gauge freedom, flux, and current loops

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Main idea. Week 5 took us from a steady current to $\mathbf{B}$ and then to magnetic forces. This week we introduce a potential for $\mathbf{B}$, reuse the Poisson-equation logic of Week 3, and describe a small current loop by one magnetic moment. We will learn about potentials and flux first, then dipoles and their motion in an applied field.

1 **A vector potential:** recover the field by taking a curl

Week 5 gave $\nabla \cdot \mathbf{B} = 0$: magnetic field lines have no sources or sinks. Locally, a smooth divergence-free field can be written as a curl. We introduce the **magnetic vector potential** $\mathbf{A}$ through

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}. \quad (1.1)$$

Unlike the electric potential Φ , $\mathbf{A}$ has three components. Its units are T m, because a spatial derivative gives a magnetic field. We work with smooth fields in the ordinary regions used below; the effect of an excluded wire or flux tube is discussed in the optional pages.

Example — a simple potential for a uniform field

Let $\mathbf{A} = B_0 x \hat{\mathbf{y}}$, with constant B_0 .

Solution. Here $A_x = A_z = 0$ and $A_y = B_0 x$. Thus

$$B_z = \partial_x A_y - \partial_y A_x = B_0, \quad B_x = B_y = 0.$$

Therefore $\boxed{\mathbf{B} = B_0 \hat{\mathbf{z}}}$, although $\mathbf{A}$ varies with x .



The field depends on the curl of $\mathbf{A}$, not on its value at one point. The green arrows increase with x ; their constant spatial rate of change gives the same blue field everywhere.

Electrostatics used $\mathbf{E} = -\nabla\Phi$. **Magnetostatics** uses $\mathbf{B} = \nabla \times \mathbf{A}$.

2 Magnetic flux: a line integral around the boundary

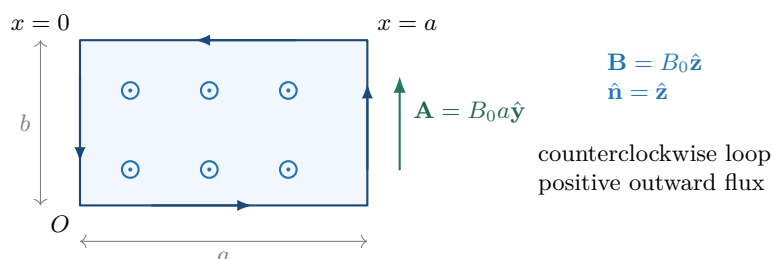
In Week 5, flux measured how much magnetic field crosses an oriented surface. Write $d\mathbf{S} = \hat{\mathbf{n}} dS$ and use Stokes' theorem:

$$\int_S \mathbf{B} \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\boldsymbol{\ell}.$$

Hence

$$\boxed{\Phi_B = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}, \quad C = \partial S.} \quad (2.1)$$

The right-hand rule links the loop direction to $\hat{\mathbf{n}}$. Here Φ_B is magnetic flux; Φ without the subscript is electric potential.



The rectangular loop is traversed counterclockwise as seen from $+z$. For this choice of potential, only the right edge contributes to its circulation.

Example — calculate the same flux in two ways

A rectangle has $0 \leq x \leq a$, $0 \leq y \leq b$, with $\mathbf{B} = B_0 \hat{\mathbf{z}}$. Use the potential from the previous page.

Solution. The direct flux is $\Phi_B = B_0 ab$. On the horizontal edges, $\mathbf{A} \cdot d\boldsymbol{\ell} = 0$. On the left edge, $A_y = 0$. On the right edge,

$$\oint_C \mathbf{A} \cdot d\boldsymbol{\ell} = \int_0^b B_0 a dy = B_0 ab.$$

For $B_0 = 0.20 \text{ T}$, $a = 0.10 \text{ m}$, and $b = 0.05 \text{ m}$,

$$\boxed{\Phi_B = 1.0 \times 10^{-3} \text{ Wb.}}$$

Reversing the loop direction reverses the signed flux.

Physical meaning

Keep the two circulation laws distinct: $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}$ counts enclosed current, whereas $\oint \mathbf{A} \cdot d\boldsymbol{\ell} = \Phi_B$ counts enclosed magnetic flux.

3 Gauge freedom: different potentials, the same field

Adding the gradient of a smooth scalar function χ leaves the curl unchanged:

$$\mathbf{A}' = \mathbf{A} + \nabla\chi. \quad (3.1)$$

Indeed, $\nabla \times \nabla\chi = 0$, so $\nabla \times \mathbf{A}' = \mathbf{B}$. This change of potential is a **gauge transformation**. It lets us choose a convenient mathematical description of a given field.

Example — two potentials for $\mathbf{B} = B_0\hat{\mathbf{z}}$

Compare the one-component (Landau) gauge and the symmetric gauge:

$$\mathbf{A}_L = B_0x\hat{\mathbf{y}}, \quad \mathbf{A}_s = \frac{B_0}{2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}).$$

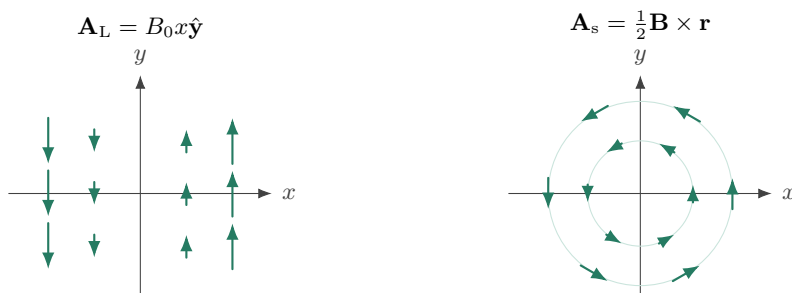
Solution. For $\mathbf{A}_L$, $B_z = B_0$. For $\mathbf{A}_s$,

$$B_z = \partial_x(B_0x/2) - \partial_y(-B_0y/2) = B_0.$$

All other field components vanish in both cases. Their difference is

$$\mathbf{A}_L - \mathbf{A}_s = \frac{B_0}{2}(y\hat{\mathbf{x}} + x\hat{\mathbf{y}}) = \nabla\left(\frac{B_0xy}{2}\right).$$

Thus the two choices are related by $\chi = B_0xy/2$.



Green arrows show two gauges for the same uniform field out of the page. Symmetry can guide our choice of $\mathbf{A}$, but it does not make that choice unique.

Check / physical lesson

For a single-valued χ , a closed path returns to the same value of χ :

$$\oint_C \nabla\chi \cdot d\ell = \chi_{\text{end}} - \chi_{\text{start}} = 0.$$

Therefore a gauge transformation changes neither $\mathbf{B}$ nor the closed-loop circulation of $\mathbf{A}$.

4 Coulomb gauge: reuse Poisson's equation

A useful choice is **Coulomb gauge**,

$$\boxed{\nabla \cdot \mathbf{A} = 0.} \quad (4.1)$$

This is a condition on the potential, separate from the physical law $\nabla \cdot \mathbf{B} = 0$. Both uniform-field gauges on the previous page satisfy it.

Turn Ampère's law into a potential equation

Insert $\mathbf{B} = \nabla \times \mathbf{A}$ into $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

The first term vanishes in Coulomb gauge. Hence

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.} \quad (4.2)$$

In Cartesian coordinates this means three scalar Poisson equations, $\nabla^2 A_i = -\mu_0 J_i$, one for each of $i = x, y, z$. Boundary or far-field conditions still select the solution.

Example — a uniform current density inside a slab

Inside $|x| < a$, let $\mathbf{J} = J_0 \hat{\mathbf{z}}$, independent of y and z . Choose $\mathbf{A} = A_z(x) \hat{\mathbf{z}}$ and a field with $B_y(0) = 0$.

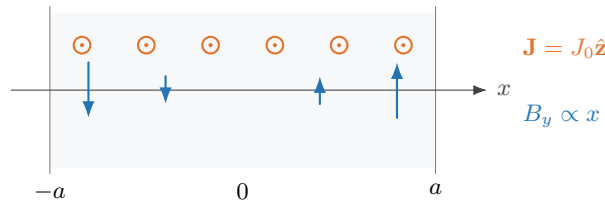
Solution. The gauge condition holds because $\partial_z A_z = 0$. Poisson's equation becomes

$$A_z'' = -\mu_0 J_0.$$

With $A_z(0) = 0$ as a reference choice and $A_z'(0) = 0$,

$$\boxed{A_z = -\frac{\mu_0 J_0 x^2}{2}}, \quad \boxed{\mathbf{B} = -A_z'(x) \hat{\mathbf{y}} = \mu_0 J_0 x \hat{\mathbf{y}}.}$$

Check. $\partial_x B_y = \mu_0 J_0$, as required by Ampère's law. These formulas describe the interior of the slab.



The field reverses across the middle of the current-carrying slab. The current points out of the page; the blue arrows follow its right-hand rule.

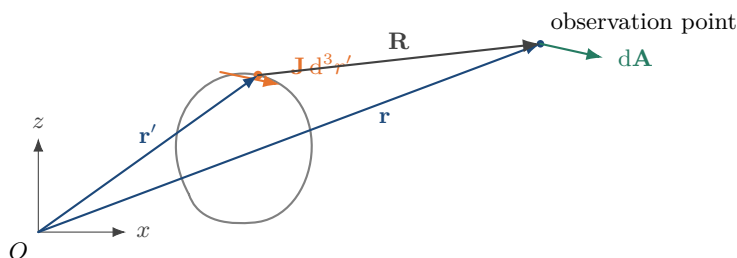
To impose Coulomb gauge on a given $\mathbf{A}$, choose χ so that $\nabla^2 \chi = -\nabla \cdot \mathbf{A}$. A further change with $\nabla^2 \chi = 0$ preserves the gauge, so the gauge condition alone does not fix $\mathbf{A}$ uniquely.

5 Known currents: integrate the vector potential

For a localized steady current in otherwise empty space, choose $\mathbf{A} \rightarrow 0$ far away. The Coulomb-gauge solution has the same $1/R$ kernel as the electric potential in Week 3:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'. \quad (5.1)$$

Here $\mathbf{r}'$ is the source point, $\mathbf{r}$ is the observation point, and $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ points from source to observation. Each small contribution to $\mathbf{A}$ is parallel to its source current.



Keep source and observation coordinates separate. When taking $\nabla \times \mathbf{A}$, differentiate with respect to $\mathbf{r}$; the source current is held fixed.

For a thin closed wire, replace $\mathbf{J} d^3r'$ by $I d\ell'$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\ell'}{|\mathbf{r} - \mathbf{r}'|}. \quad (5.2)$$

For a current sheet, the replacement is $\mathbf{J} d^3r' \rightarrow \mathbf{K} dS'$. The current density $\mathbf{K}$ has units A/m, as in Week 5.

Example — zero potential at the centre of a current loop

A circular loop of radius R carries current I . Find $\mathbf{A}$ at its centre in the source-integral gauge.

Solution. All source points are distance R from the centre, so

$$\mathbf{A}(0) = \frac{\mu_0 I}{4\pi R} \oint_C d\ell' = 0.$$

The displacement vectors around a closed path add to zero. Yet Week 5 gave $\mathbf{B}(0) = \mu_0 I \hat{\mathbf{z}} / (2R)$ for the chosen current orientation. A zero value of $\mathbf{A}$ at one point does not imply a zero curl there.

Method / strategy

For a known localized current: integrate $\mathbf{A}$, then take its curl. If the field is already known from symmetry, it may be easier to construct $\mathbf{A}$ directly. The infinite wire on the next page uses this second route.

6 An infinite wire: integrate the known field

An infinite straight wire carries current I along $+z$. Let ϱ be the distance from its axis and $\hat{\phi}$ the direction around it. Week 5 gave

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\varrho} \hat{\phi}.$$

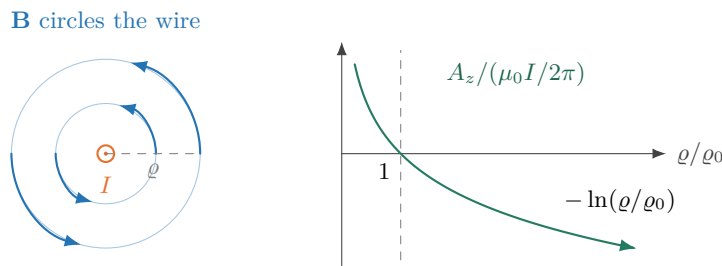
Choose $\mathbf{A} = A_z(\varrho)\hat{\mathbf{z}}$. For this form, the cylindrical-coordinate curl is

$$B_\phi = -\frac{dA_z}{d\varrho}.$$

Thus $-A'_z = \mu_0 I / (2\pi\varrho)$, and integration gives

$$\boxed{\mathbf{A} = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{\varrho}{\varrho_0}\right) \hat{\mathbf{z}}.} \quad (6.1)$$

The reference radius $\varrho_0 > 0$ fixes where A_z is zero. The logarithm has a dimensionless argument.



The physical field circles the wire. In this gauge the potential points along the wire, and its negative radial slope gives B_ϕ . The zero of A_z is a reference choice.

Example — double the observation radius

Let $I = 5.0$ A. Compare A_z and B_ϕ at ϱ_0 and $2\varrho_0$.

Solution. Using $\mu_0 \simeq 4\pi \times 10^{-7}$ T m/A,

$$A_z(2\varrho_0) - A_z(\varrho_0) = -\frac{\mu_0 I}{2\pi} \ln 2 \simeq -6.9 \times 10^{-7} \text{ T m}.$$

Meanwhile $B_\phi(2\varrho_0) = B_\phi(\varrho_0)/2$. Changing ϱ_0 shifts A_z by a constant but leaves this field unchanged.

Check / physical lesson

$\nabla \cdot \mathbf{A} = \partial_z A_z = 0$, so this is Coulomb gauge. The infinite wire is not a localized source: its logarithmic potential cannot obey $\mathbf{A} \rightarrow 0$ at large radial distance.

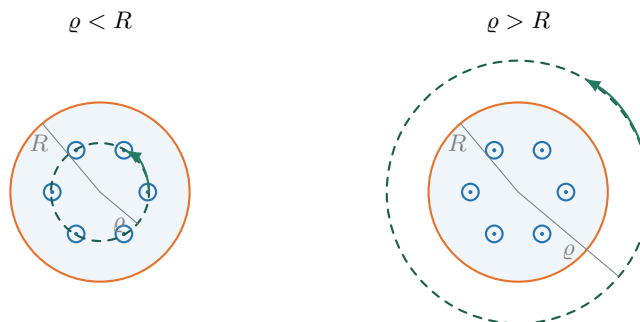
7 A solenoid: use flux to find the potential

Take an ideal infinitely long solenoid of radius R . Its field is $\mathbf{B} = B_0 \hat{\mathbf{z}}$ inside and zero outside. Week 5 gives $B_0 = \mu_0 n I$, where n is the number of turns per unit length. Choose a rotationally symmetric potential $\mathbf{A} = A_\phi(\varrho) \hat{\phi}$.

On a circle of radius ϱ , this potential is tangent and has constant magnitude:

$$\oint_C \mathbf{A} \cdot d\boldsymbol{\ell} = 2\pi\varrho A_\phi.$$

Set this equal to the flux through the circle.



Blue dots mark the field inside the solenoid; it points out of the page. The outer loop encloses more area, but the field-filled area stops growing at R . The green circles are the paths used for the circulation of $\mathbf{A}$.

Example — inside and outside, one flux rule

Find A_ϕ on both sides of the solenoid surface.

Solution. For $\varrho < R$, the enclosed flux is $B_0 \pi \varrho^2$:

$$2\pi\varrho A_\phi = B_0 \pi \varrho^2.$$

For $\varrho > R$, the flux is the constant $\Phi_B = B_0 \pi R^2$:

$$2\pi\varrho A_\phi = B_0 \pi R^2.$$

Therefore

$$A_\phi(\varrho) = \begin{cases} B_0 \varrho / 2, & 0 \leq \varrho \leq R, \\ B_0 R^2 / (2\varrho), & \varrho \geq R. \end{cases} \quad (7.1)$$

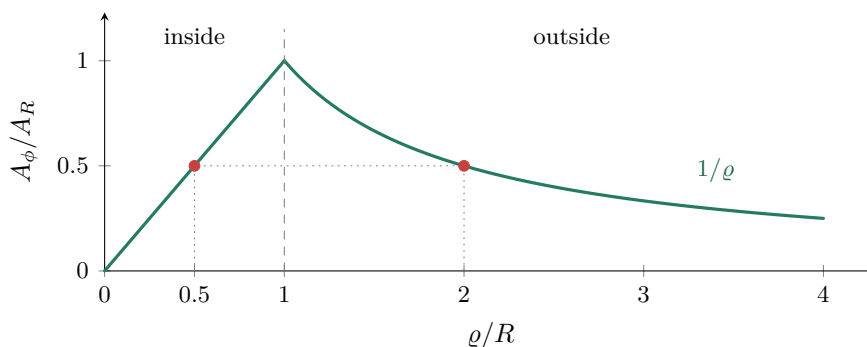
Physical meaning

Outside, $\mathbf{B} = 0$ but this $\mathbf{A}$ is nonzero. Its circulation around the solenoid records the flux inside. A gauge change with a single-valued χ preserves that circulation.

7.1 Read and check the solenoid potential

The potential rises linearly inside and falls as $1/\varrho$ outside. Both expressions agree at the surface:

$$A_\phi(R) = A_R \equiv \frac{B_0 R}{2}.$$



The potential is continuous at R , but its slope changes. The marked radii have the same A_ϕ , even though their magnetic fields differ.

Take the curl to recover the field

For $\mathbf{A} = A_\phi(\varrho)\hat{\phi}$, independent of z and ϕ ,

$$B_z = \frac{1}{\varrho} \frac{d}{d\varrho}(\varrho A_\phi).$$

Inside, $\varrho A_\phi = B_0 \varrho^2/2$, so $B_z = B_0$. Outside, $\varrho A_\phi = B_0 R^2/2$ is constant, so $B_z = 0$. This verifies both branches of Eq. (7.1).

Example — equal potential values do not mean equal fields

Let $R = 2.0$ cm and $B_0 = 0.10$ T. Compare the radii $R/2$ and $2R$.

Solution. $A_R = B_0 R/2 = 1.0 \times 10^{-3}$ T m, so

$$A_\phi(R/2) = A_\phi(2R) = 5.0 \times 10^{-4} \text{ T m}.$$

Yet $B_z(R/2) = 0.10$ T and $B_z(2R) = 0$. The outer circulation is four times the inner one because its path is four times longer.

Check / physical lesson

The jump in the tangential field B_z at R is supported by the solenoid's surface current, as in Week 5. It is compatible with a continuous A_ϕ . The potential's spatial derivatives determine the field.

We now turn from extended currents to a small current loop seen from far away.

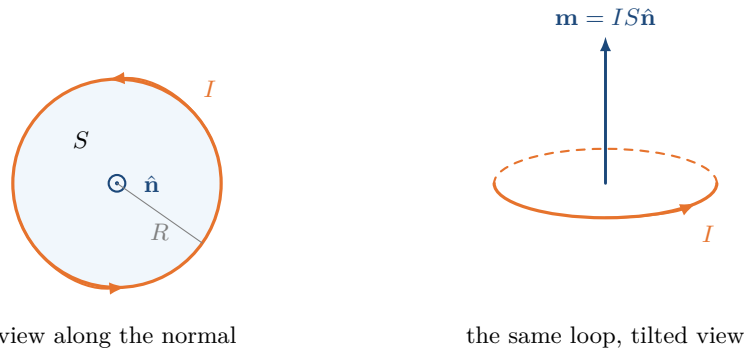
8 A current loop: define its magnetic moment

From a small loop to a magnetic dipole

We now describe the strength and orientation of a planar current loop by one vector. If the loop has area S and carries current I , its **magnetic dipole moment** is

$$\mathbf{m} = IS\hat{\mathbf{n}}. \quad (8.1)$$

Curl the fingers of your right hand with the current; your thumb gives $\hat{\mathbf{n}}$. For N identical turns carrying the same current, $\mathbf{m} = NIS\hat{\mathbf{n}}$. The units of m are A m^2 .



The current fixes the direction of the area normal and of $\mathbf{m}$. Reversing the current reverses both vectors. The letter S denotes area; $\mathbf{A}$ denotes vector potential.

Example — the small coil used below

A coil has $N = 50$ turns, radius $R = 2.0 \text{ cm}$, and current $I = 0.20 \text{ A}$. Find its moment.

Solution. Each turn encloses $S = \pi R^2 = 1.26 \times 10^{-3} \text{ m}^2$. Thus

$$m = NI\pi R^2 = 1.26 \times 10^{-2} \text{ A m}^2.$$

Its direction follows the right-hand rule. We will use this same coil in the examples on torque, energy, and force.

Physical meaning

The moment combines three ways to strengthen a coil: increase the current, the area of each turn, or the number of turns. It does not replace the detailed current pattern close to the wire.

At distances much larger than the loop, its leading field depends only on $\mathbf{m}$.

9 Far from a loop: keep the first nonzero term

We use the same large-distance expansion as in Week 4. Let a be the size of a localized loop, and observe it at $r \gg a$. Write $\hat{\mathbf{r}} = \mathbf{r}/r$.

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r^2} + \dots$$

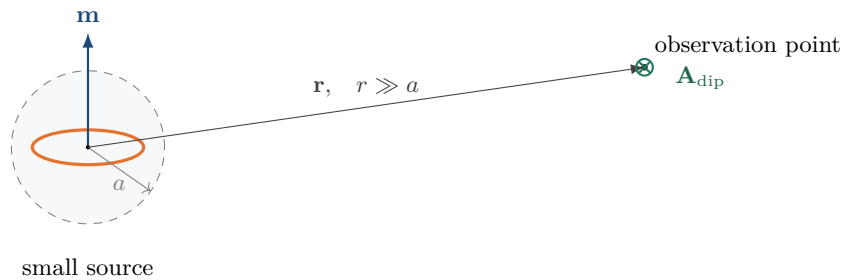
Substituting this into the wire formula gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint_C d\ell' + \frac{1}{r^2} \oint_C (\hat{\mathbf{r}} \cdot \mathbf{r}') d\ell' + \dots \right].$$

The first integral vanishes: a closed path has zero total displacement. The next integral is $S\hat{\mathbf{n}} \times \hat{\mathbf{r}}$ for a planar loop. Therefore

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \quad (9.1)$$

This is the **dipole approximation**: keep the leading term of a small source as seen from far away. The last optional page derives the moment formula for any localized steady current.



Far away, the details of the wire become small compared with the observation distance. At the point shown, $\mathbf{A}_{\text{dip}}$ points into the page, perpendicular to the plane of $\mathbf{m}$ and $\mathbf{r}$.

Example — potential in the equatorial plane

A loop has $\mathbf{m} = m\hat{\mathbf{z}}$. Find the dipole potential at $(r, 0, 0)$, with $r \gg a$.

Solution. Here $\hat{\mathbf{r}} = \hat{\mathbf{x}}$ and $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$, so

$$\mathbf{A}_{\text{dip}} = \frac{\mu_0 m}{4\pi r^2} \hat{\mathbf{y}}.$$

Doubling r reduces this potential by a factor of four, in this gauge.

Check / physical lesson

The dipole is the leading term *when* $\mathbf{m} \neq 0$. Two nearby loops with equal and opposite moments have zero total dipole moment; the next nonzero multipole then determines their far field.

10 The dipole field: take one curl

Choose the polar axis along $\mathbf{m} = m\hat{\mathbf{z}}$. The angle θ is measured from $+z$. Since $\mathbf{m} \times \hat{\mathbf{r}} = m \sin \theta \hat{\boldsymbol{\phi}}$, Eq. (9.1) becomes

$$\mathbf{A}_{\text{dip}} = A_{\phi} \hat{\boldsymbol{\phi}}, \quad A_{\phi} = \frac{\mu_0 m \sin \theta}{4\pi r^2}.$$

Curl in spherical coordinates

Only A_{ϕ} is nonzero, and nothing depends on ϕ . Hence

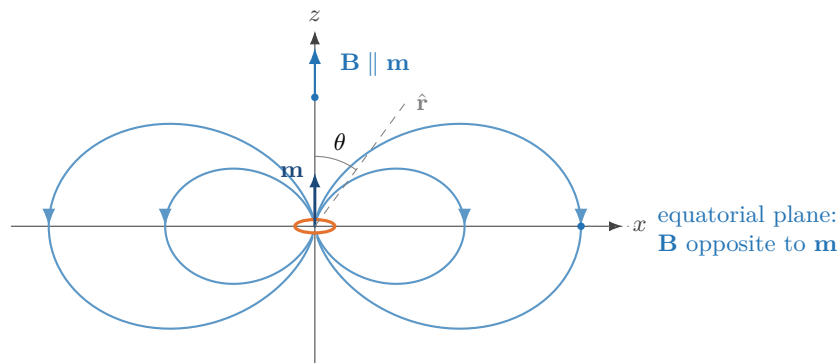
$$B_r = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_{\phi}) = \frac{\mu_0}{4\pi} \frac{2m \cos \theta}{r^3},$$

$$B_{\theta} = -\frac{1}{r} \frac{\partial}{\partial r} (r A_{\phi}) = \frac{\mu_0}{4\pi} \frac{m \sin \theta}{r^3}, \quad B_{\phi} = 0.$$

Using $\hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}$ combines these components into one vector.

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]. \quad (10.1)$$

This field falls as $1/r^3$. For a finite loop it applies at $r \gg a$; the ideal point-dipole expression is used only away from the origin.



Dipole field lines in a plane through the axis. The field points along $\mathbf{m}$ on the axis and opposite to it in the equatorial plane.

Example — axis versus equator at the same distance

Compare the two fields a distance r from a dipole $m\hat{\mathbf{z}}$.

Solution. On the positive axis, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$; at the equator, $\hat{\mathbf{r}} \cdot \mathbf{m} = 0$:

$$\mathbf{B}_{\text{axis}} = \frac{\mu_0}{4\pi} \frac{2m}{r^3} \hat{\mathbf{z}}, \quad \mathbf{B}_{\text{equator}} = -\frac{\mu_0}{4\pi} \frac{m}{r^3} \hat{\mathbf{z}}.$$

The axial field is twice as strong and points in the opposite direction. This angular pattern is the same as the electric dipole in Week 4.

11 Check the approximation: an exact circular loop

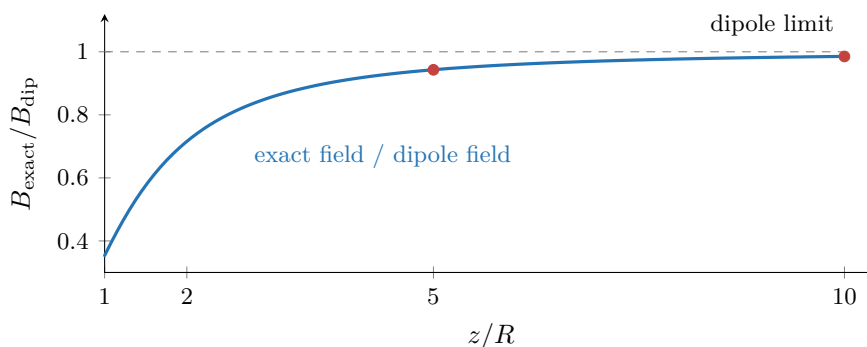
We can test the dipole limit against a field already known from Week 5. A circular loop of radius R and current I has, on its axis,

$$\mathbf{B}_{\text{exact}}(z) = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}. \quad (11.1)$$

For $z \gg R$, factor out z^3 from the denominator:

$$B_{\text{exact}} = \frac{\mu_0 I R^2}{2z^3} \left(1 + \frac{R^2}{z^2}\right)^{-3/2} \simeq \frac{\mu_0 I R^2}{2z^3}.$$

With $m = I\pi R^2$, this is exactly the axial dipole field, $B_{\text{dip}} = \mu_0(2m)/(4\pi z^3)$.



The ratio approaches one as the observation point moves away from the loop. The dipole expression overestimates the exact axial field at every finite $z > 0$.

Example — how far is far enough?

Estimate the error in replacing the loop by a dipole at $z = 5R$ and $z = 10R$.

Solution. The fractional overestimate, measured relative to the exact field, is

$$\frac{B_{\text{dip}} - B_{\text{exact}}}{B_{\text{exact}}} = \left(1 + \frac{R^2}{z^2}\right)^{3/2} - 1.$$

At $5R$ it is 6.1%; at $10R$ it is 1.5%. Thus the acceptable distance depends on the accuracy required.

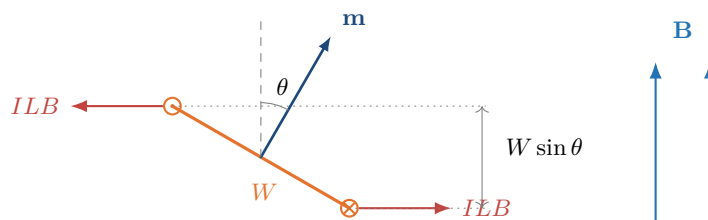
Check / physical lesson

Near the centre, the exact field is finite: $B(0) = \mu_0 I / (2R)$. The far-field formula cannot be extrapolated to $z = 0$. For this symmetric loop, the first relative correction scales as $(R/z)^2$.

The dipole describes both the field produced by a small loop and its response to an applied field.

12 Torque: a uniform field turns the loop

We now find how an *applied* field acts on the loop. Throughout this section, $\mathbf{B}$ means the external field, not the loop's own field. In a uniform field the net force is zero, but forces on opposite sides can form a **couple**: equal and opposite forces whose lines of action are separated.



side view: the length- L sides run into and out of the page

A rectangular loop tilted by θ . Orange symbols show opposite currents. The red forces make a couple that turns $\mathbf{m}$ toward the blue field.

A force couple gives the torque

Take a rectangle of sides L and W , tilted as shown, with the L -sides perpendicular to $\mathbf{B}$. Each of these sides feels a force ILB . Their perpendicular separation is $W \sin \theta$, so

$$\tau = (ILB)(W \sin \theta) = I(LW)B \sin \theta = mB \sin \theta.$$

The forces on the other pair act along lines through the centre and give no torque; they need not vanish. The torque direction follows the cross product below.

$$\boxed{\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}, \quad \tau = mB \sin \theta.} \quad (12.1)$$

The same result holds for a planar loop of any shape in a uniform field, and for a small dipole when the field varies little over its size.

Example — torque on our coil

Use the coil from Sec. 8, with $m = 1.26 \times 10^{-2} \text{ A m}^2$. Let $B = 0.30 \text{ T}$ and $\theta = 60^\circ$.

Solution.

$$\tau = mB \sin 60^\circ \simeq 3.3 \times 10^{-3} \text{ N m}.$$

The instantaneous torque tends to reduce θ . It is zero at both 0° and 180° , and largest at 90° .

Check / physical lesson

For a closed loop in a uniform field, $\mathbf{F} = I \oint d\boldsymbol{\ell} \times \mathbf{B} = \mathbf{0}$. Zero net force does not imply zero torque.

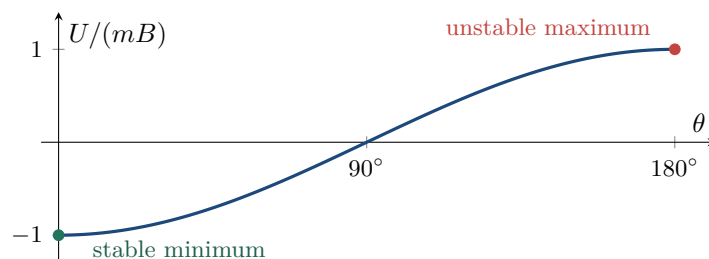
13 Energy: which orientation is stable?

Torque tells us how the loop starts to turn. Energy tells us which orientations are stable and how much mechanical work the field can do. For fixed m in a uniform applied field, the torque that changes θ is $\tau_\theta = -mB \sin \theta$. Using $\tau_\theta = -dU/d\theta$ gives

$$\frac{dU}{d\theta} = mB \sin \theta \quad \Rightarrow \quad U = -mB \cos \theta + \text{constant}.$$

Choose $U = 0$ at $\theta = 90^\circ$. Then

$$U = -\mathbf{m} \cdot \mathbf{B} = -mB \cos \theta. \quad (13.1)$$



Parallel alignment has the lowest energy. Antiparallel alignment has zero torque but is unstable: a small angular displacement produces a torque away from it.

Example — work as the coil turns toward the field

Our coil turns from 60° to 0° in $B = 0.30$ T. Find the change in energy and the work done by the magnetic torque.

Solution.

$$\Delta U = -mB(\cos 0^\circ - \cos 60^\circ) = -\frac{mB}{2} \simeq -1.9 \times 10^{-3} \text{ J}.$$

Thus the torque does $W = -\Delta U \simeq 1.9$ mJ of mechanical work. If the coil is free and starts from rest, this becomes rotational kinetic energy as it first reaches alignment.

Physical meaning

The torque tends to align the moment, but a freely rotating coil need not stop there. Without damping it can overshoot and oscillate. Damping allows it to settle at the energy minimum.

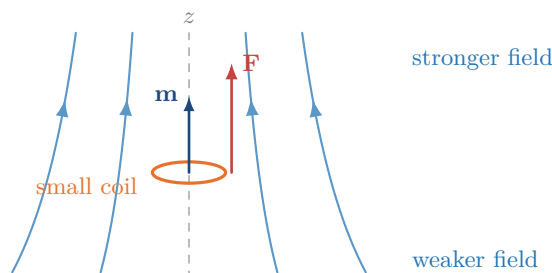
Here U is the mechanical interaction energy for a fixed magnetic moment, including a small loop whose current is maintained. The current source can exchange energy with the system. An induced moment that changes with $\mathbf{B}$ requires a separate energy calculation.

14 Force: a field gradient moves the dipole

A uniform field can turn the loop but cannot pull it as a whole. In a slowly varying applied field, opposite sides feel slightly different forces. For a small dipole with fixed $\mathbf{m}$, use the position-dependent energy:

$$\mathbf{F} = -\nabla U = \nabla(\mathbf{m} \cdot \mathbf{B}). \quad (14.1)$$

The gradient is taken with the dipole's orientation held fixed. The field must vary little across the loop, so one moment remains an adequate description.



The field lines converge as the axial field grows. An aligned, fixed moment is pulled toward larger B_z . The drawing shows the local field used below.

Example — an aligned coil near the axis of a magnet

Near the origin, take the applied field to be

$$\mathbf{B} = -\frac{Gx}{2}\hat{\mathbf{x}} - \frac{Gy}{2}\hat{\mathbf{y}} + (B_0 + Gz)\hat{\mathbf{z}}, \quad B_0 > 0, \quad G > 0.$$

On the z -axis, the field points along $+z$ and grows with z . Place our small coil on this axis with $\mathbf{m} = m\hat{\mathbf{z}}$. Find its force.

Solution. The dot product is $\mathbf{m} \cdot \mathbf{B} = m(B_0 + Gz)$, so

$$\mathbf{F} = mG\hat{\mathbf{z}}.$$

For $G = 2.0 \text{ T/m}$ and $m = 1.26 \times 10^{-2} \text{ A m}^2$, $F \simeq 2.5 \times 10^{-2} \text{ N}$, toward the stronger field. The torque is zero on the axis because $\mathbf{m} \parallel \mathbf{B}$.

Check / physical lesson

This local field obeys $\nabla \cdot \mathbf{B} = -G/2 - G/2 + G = 0$ and $\nabla \times \mathbf{B} = \mathbf{0}$. Its transverse components are essential: a field written as $B(z)\hat{\mathbf{z}}$ everywhere cannot satisfy $\nabla \cdot \mathbf{B} = 0$ when $B'(z) \neq 0$.

Torque responds to orientation; net force responds to spatial variation.

15 Choose the shortest route: from sources to motion

The useful method depends on what is known. The following choices connect this week to the field calculations of Week 5 and the potential methods of Weeks 3 and 4.

Method / strategy

Starting information

A direct route

A known localized current	Integrate $\mathbf{A}$ from the source, then take its curl.
A known symmetric field	Choose a symmetry-compatible form of $\mathbf{A}$ and integrate a curl component or the enclosed flux.
A current density and boundary data	Use Coulomb gauge and solve $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ component by component.
A small loop, viewed far away	Find $\mathbf{m}$ and use the dipole potential or field.
A small loop in an applied field	Use torque for rotation, energy for stability, and a field gradient for translation.

Example — one source, two observation regimes

You know the radius and current of a circular loop. You need its field at its centre and on its axis very far away. Which method should you use?

Solution. At the centre, use the exact Biot–Savart result from Week 5, $B = \mu_0 I / (2R)$. Far away, use $m = I\pi R^2$ and $B = \mu_0(2m)/(4\pi r^3)$. The dipole method saves work only in its range of validity.

Reference	Relation and condition
Potential and flux	$\mathbf{B} = \nabla \times \mathbf{A}; \quad \Phi_B = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}$ for a spanning surface where $\mathbf{A}$ is defined.
Gauge	$\mathbf{A}' = \mathbf{A} + \nabla\chi$; smooth, single-valued χ preserves closed circulation.
Coulomb gauge	$\nabla \cdot \mathbf{A} = 0, \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ in magnetostatics.
Moment	$\mathbf{m} = NIS\hat{\mathbf{n}}$ for identical planar turns; $\mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J} d^3r'$ for a localized steady current.
Dipole field	$\mathbf{B} = \mu_0[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]/(4\pi r^3), \quad r \gg \text{source size.}$
Response to an applied field	$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}, \quad U = -\mathbf{m} \cdot \mathbf{B}, \quad \mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$ for fixed $\mathbf{m}$.

15.1 Bridge to Week 7: when the flux changes

So far the currents and fields have been steady. For a fixed loop, Faraday’s law gives

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt} = -\oint_C \frac{\partial \mathbf{A}}{\partial t} \cdot d\boldsymbol{\ell}.$$

This leads to the time-dependent potential form

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Week 7 develops this new contribution to the electric field. Under a time-dependent gauge change, $\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$ must be accompanied by $\Phi \rightarrow \Phi - \partial_t\chi$ to keep $\mathbf{E}$ unchanged.

The main development ends here. The next four pages give optional proofs and global details.

OPTIONAL — Why the source integral works

Further detail, following the main development.

16 Check the source formula: Poisson, gauge, and curl

This page checks the potential used in Sec. 5. Assume a localized steady current: $\nabla' \cdot \mathbf{J} = 0$, with no current at a distant boundary. Set $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $G = 1/R$. Here ∇ acts on $\mathbf{r}$, while ∇' acts on $\mathbf{r}'$.

1. The same Poisson kernel as in electrostatics

The identity $\nabla^2 G = -4\pi\delta^{(3)}(\mathbf{r} - \mathbf{r}')$ means that the Laplacian of $1/R$ vanishes away from its source and supplies a point source there. The delta function selects the value of the integrand at $\mathbf{r}' = \mathbf{r}$. Therefore

$$\nabla^2 \mathbf{A} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \nabla^2 G \, d^3 r' = -\mu_0 \mathbf{J}(\mathbf{r}).$$

Together with decay at infinity, this is the free-space solution of the vector Poisson equation.

2. Steady current makes this potential a Coulomb-gauge potential

Use $\nabla G = -\nabla' G$ and integrate by parts:

$$\begin{aligned} \nabla \cdot \mathbf{A} &= -\frac{\mu_0}{4\pi} \int \mathbf{J} \cdot \nabla' G \, d^3 r' \\ &= -\frac{\mu_0}{4\pi} \oint_{\partial V'} G \mathbf{J} \cdot \hat{\mathbf{n}}' \, dS' + \frac{\mu_0}{4\pi} \int G \nabla' \cdot \mathbf{J} \, d^3 r' = 0. \end{aligned}$$

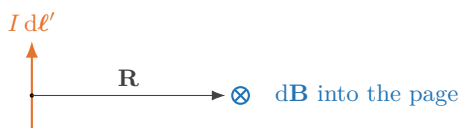
Localization removes the surface term; steady charge conservation removes the volume term. These are two separate assumptions.

3. Taking the curl recovers Biot–Savart

The current depends only on the source point, so $\nabla \times \mathbf{J}(\mathbf{r}') = 0$. Since $\nabla G = -\mathbf{R}/R^3$,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \nabla G \times \mathbf{J} \, d^3 r' = \boxed{\frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \mathbf{R}}{R^3} \, d^3 r'}.$$

The minus sign is removed by reversing the order of the cross product. For a thin wire, replace $\mathbf{J} \, d^3 r'$ by $I \, d\ell'$.



A sign check: an upward current element and a rightward displacement produce a magnetic-field contribution into the page, as in Week 5.

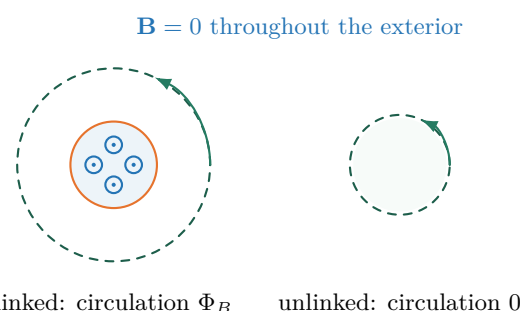
OPTIONAL — A field-free region can still enclose flux
Further detail, following the main development.

17 Outside a solenoid: local curl and global circulation

We now explain why the exterior potential cannot simply be set to zero everywhere. For the ideal solenoid, $\mathbf{B} = 0$ outside, but a loop that goes once around the solenoid has

$$\oint_C \mathbf{A} \cdot d\boldsymbol{\ell} = \Phi_B, \quad \Phi_B = B_0 \pi R^2.$$

The word **local** concerns a small neighbourhood. **Global** concerns the whole region, including paths that surround its excluded interior.



The left path surrounds the flux-filled interior; the right path bounds a disk entirely outside it. The excluded solenoid makes a hole in the region where we are using $\mathbf{B} = 0$.

17.1 Stokes' theorem needs a whole spanning surface

For the unlinked loop, a spanning disk lies in the field-free exterior, so its flux is zero. Every surface spanning the linked loop intersects the flux tube. We cannot use $\nabla \times \mathbf{A} = 0$ on that whole surface. There is no conflict with Stokes' theorem.

Example — go twice around the solenoid

For the solenoid of Sec. 7, $R = 2.0$ cm and $B_0 = 0.10$ T. What is the circulation for two positive circuits?

Solution. $\Phi_B = B_0 \pi R^2 = 1.26 \times 10^{-4}$ Wb, so the answer is $2\Phi_B = 2.51 \times 10^{-4}$ Wb. More generally, the circulation is $k\Phi_B$, where the integer k counts the signed number of turns around the solenoid. Deforming the path without crossing the solenoid does not change k .

Why a gauge can remove the potential locally, but not on the whole exterior

Outside, $\mathbf{A} = [\Phi_B/(2\pi\varrho)]\hat{\phi}$ and $\nabla\phi = \hat{\phi}/\varrho$. On a patch where the angle ϕ is single-valued, choose

$$\chi = -\frac{\Phi_B}{2\pi}\phi, \quad \mathbf{A} + \nabla\chi = \mathbf{0}.$$

After one full circuit, however, χ changes by $-\Phi_B$ at the same point. It is not a single-valued gauge function on the whole exterior. A valid global gauge change cannot remove the nonzero closed circulation.

Quantum connection: the Aharonov–Bohm effect links this enclosed flux to an interference phase, even when the particle paths remain in a field-free region. The quantum theory is beyond this week's scope.

OPTIONAL — When a magnetic scalar potential is useful

Further detail, following the main development.

18 Current-free space: a second potential description

We now find the condition for using a scalar potential for $\mathbf{B}$. In a current-free magnetostatic region, Ampère's law gives $\nabla \times \mathbf{B} = 0$. In a small simply connected patch, we may therefore write

$$\boxed{\mathbf{B} = -\nabla\Psi.} \quad (18.1)$$

Here **simply connected** means that every closed path can shrink to a point without leaving the region. With $\nabla \cdot \mathbf{B} = 0$, we get

$$\nabla^2\Psi = 0.$$

Thus the Laplace-equation methods of Weeks 3 and 4 can also describe current-free magnetic regions. With this definition, Ψ has units T m.

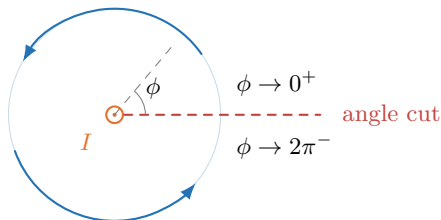
Example — the scalar potential outside a straight wire

For current I along $+z$, $\mathbf{B} = \mu_0 I \hat{\phi} / (2\pi \varrho)$. Use $\nabla\phi = \hat{\phi}/\varrho$ to find Ψ .

Solution. On any patch with a continuous choice of the angle,

$$\Psi = -\frac{\mu_0 I}{2\pi} \phi + \text{constant} \quad \Rightarrow \quad -\nabla\Psi = \frac{\mu_0 I}{2\pi \varrho} \hat{\phi}.$$

After a positive circuit, $\Delta\Psi = -\mu_0 I$. So Ψ cannot be single-valued on the whole exterior of the wire. The linked circulation $\oint \mathbf{B} \cdot d\ell = \mu_0 I$ is the obstruction.



cut the domain to choose one continuous branch of ϕ

The cut is a bookkeeping choice for the angle, not a physical discontinuity in $\mathbf{B}$. The values of Ψ on its two sides differ by $\mu_0 I$.

Physical meaning

The scalar potential Ψ is useful where there is no current. The vector potential $\mathbf{A}$ describes regions with current as well. They are different constructions: $\mathbf{B} = -\nabla\Psi$ versus $\mathbf{B} = \nabla \times \mathbf{A}$. In both cases, a local derivative relation must be distinguished from circulation around a hole in the region.

OPTIONAL — The magnetic moment of a general localized current
Further detail, following the main development.

19 Complete the dipole derivation: one conservation rule

We now justify the far-field moment formula without assuming a planar loop. For a localized steady current, integration by parts and $\nabla' \cdot \mathbf{J} = 0$ give

$$\int \mathbf{J} \cdot \nabla' f \, d^3r' = \oint f \mathbf{J} \cdot \hat{\mathbf{n}}' \, dS' - \int f \nabla' \cdot \mathbf{J} \, d^3r' = 0.$$

This holds for the coordinate functions used below; the boundary lies outside the source.

1. The first term in the far-field expansion vanishes

Set $f = x'_i$, one Cartesian coordinate. Since $\nabla' x'_i = \hat{\mathbf{e}}_i$, $\int J_i \, d^3r' = 0$ for each component, and hence

$$\int \mathbf{J} \, d^3r' = \mathbf{0}.$$

There is no $1/r$ term in the source-integral potential of a localized steady current.

2. Identify the next term with a single vector

Let $\mathbf{u}$ be any constant vector and set $f = x'_i(\mathbf{u} \cdot \mathbf{r}')$. Applying the same rule to each component gives the vector identity

$$\int [\mathbf{r}'(\mathbf{u} \cdot \mathbf{J}) + (\mathbf{u} \cdot \mathbf{r}')\mathbf{J}] \, d^3r' = \mathbf{0}.$$

Define

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \, d^3r'. \quad (19.1)$$

The vector triple product and the identity above then give

$$\mathbf{m} \times \mathbf{u} = \frac{1}{2} \int [\mathbf{J}(\mathbf{r}' \cdot \mathbf{u}) - \mathbf{r}'(\mathbf{J} \cdot \mathbf{u})] \, d^3r' = \int (\mathbf{u} \cdot \mathbf{r}')\mathbf{J} \, d^3r'.$$

Choose $\mathbf{u} = \hat{\mathbf{r}}$ in the $1/r^2$ term of the source expansion. This yields $\mathbf{A}_{\text{dip}} = \mu_0(\mathbf{m} \times \hat{\mathbf{r}})/(4\pi r^2)$, as in Sec. 9.

Example — recover the planar-loop formula

For a thin loop, $\mathbf{m} = (I/2) \oint \mathbf{r}' \times d\boldsymbol{\ell}'$. For a circle in the xy -plane, $\mathbf{r}' = R(\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}})$ and $d\boldsymbol{\ell}' = R(-\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}) d\phi$.

Solution. Their cross product is $R^2 \hat{\mathbf{z}} d\phi$. Thus

$$\mathbf{m} = \frac{I}{2} \int_0^{2\pi} R^2 \hat{\mathbf{z}} d\phi = I\pi R^2 \hat{\mathbf{z}}.$$

For any planar loop, $\frac{1}{2} \oint \mathbf{r}' \times d\boldsymbol{\ell}' = S\hat{\mathbf{n}}$, so the same calculation gives $\mathbf{m} = IS\hat{\mathbf{n}}$.

Check / physical lesson

The moment is independent of the coordinate origin. If $\mathbf{r}' \rightarrow \mathbf{r}' - \mathbf{a}$, then $\mathbf{m} \rightarrow \mathbf{m} - \frac{1}{2}\mathbf{a} \times \int \mathbf{J} \, d^3r' = \mathbf{m}$. Steady current conservation is what makes this source property well defined.