

Electrodynamics & Relativity

Week 7: Electromagnetic Induction

Charge conservation and Maxwell's equations

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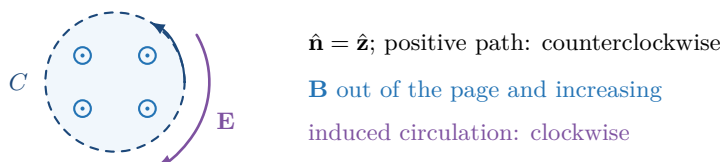
Main idea. Week 6 described magnetic flux using a vector potential. We now let the flux change. First we calculate induced electric fields and emf in fixed and moving circuits. Then charge conservation leads us to Maxwell's correction and the complete field equations. The same solenoid, moving rod, and capacitor will prepare us for Week 8's potentials and field conservation. Four optional pages develop the main ideas further.

1 Changing flux: a new source of electric circulation

We first calculate the emf around a fixed loop. Week 6 gave $\Phi_B = \int_S \mathbf{B} \cdot \hat{\mathbf{n}} dS = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}$, where C is the boundary of S . For a *fixed* loop and surface, **Faraday's law** says

$$\mathcal{E} \equiv \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}. \quad (1.1)$$

The loop direction and the surface normal are linked by the right-hand rule. The circulation $\mathcal{E}$ is called the **electromotive force**, or emf. It is measured in volts: $1 \text{ Wb/s} = 1 \text{ V}$.



The changing field produces electric circulation even if no wire is present. Blue shows $\mathbf{B}$; violet shows the induced $\mathbf{E}$.

Example — a field ramp with an arbitrary rate

A uniform perpendicular field is $\mathbf{B} = (B_0 + \alpha t)\hat{\mathbf{z}}$ over a fixed loop of area S_0 . Find the emf and its integral over a time T .

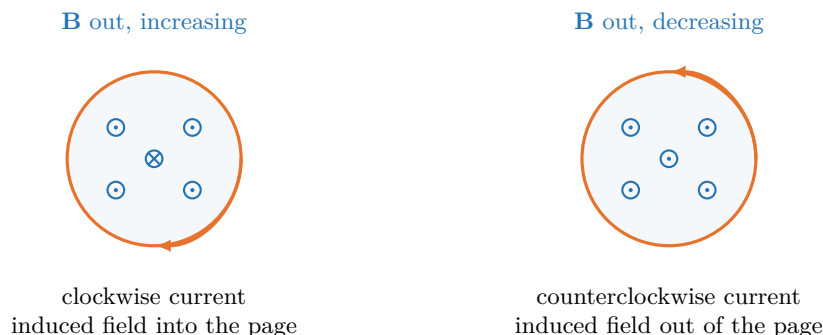
Solution. With the positive normal $+\hat{\mathbf{z}}$,

$$\Phi_B = S_0(B_0 + \alpha t), \quad \mathcal{E} = -S_0\alpha, \quad \int_0^T \mathcal{E} dt = -S_0\alpha T.$$

The initial field B_0 drops out. For $\alpha > 0$, the emf is clockwise; for $\alpha < 0$, it reverses. The time integral is $-S_0\Delta B$.

2 Lenz's law: the sign tells us the response

We now translate a signed emf into a physical current direction. The minus sign in Faraday's law is **Lenz's law**. For the resistive loop considered here, the induced current produces a field that opposes the applied flux change. We neglect the loop's own changing flux.



The applied field has the same direction in both cases. The orange current reverses when the *rate of change* reverses. The central symbol shows the induced field direction.

Method / strategy

Choose a normal and its positive loop direction. Decide whether the *signed* flux increases or decreases. Use $\mathcal{E} = -\dot{\Phi}_B$. Then use the right-hand rule to check the induced current's magnetic field.

Example — transported charge depends on the flux change

Give the fixed loop area S_0 and resistance $\mathcal{R}$. The uniform field changes smoothly from B_i to B_f . Neglect self-inductance. Find the signed current and the net charge Q_{pass} that passes a point of the wire.

Solution. In the positive counterclockwise direction,

$$I(t) = -\frac{S_0}{\mathcal{R}} \dot{B}(t), \quad Q_{\text{pass}} = \int_{t_i}^{t_f} I(t) dt = -\frac{S_0}{\mathcal{R}} (B_f - B_i).$$

The current depends on the ramp rate, but its time integral depends only on the end fields. For a linear ramp, $I = -S_0\alpha/\mathcal{R}$ throughout the ramp.

Physical meaning

Lenz's law opposes the *change*, not always the applied field itself. When an outward field decreases, the induced response supports the remaining outward flux. A constant applied flux gives no external induction emf; a current already stored in an inductive circuit can still persist or decay.

2.1 Emf is an energy-per-charge quantity

Despite its name, emf is not a force: a force has units of newtons, while $\mathcal{E}$ has units of joules per coulomb. For a stationary circuit it is the line integral of electric force per unit charge around the complete path. An open circuit can have induced electric fields without a sustained circulating current.

3 A changing solenoid: calculate the electric field

Reuse the long solenoid of Week 6, with radius R and axial field $\mathbf{B} = B(t)\hat{\mathbf{z}}$ inside. Neglect the exterior magnetic field. We calculate its circulating electric field, with no added electrostatic field. For a solenoid with n turns per unit length, $B(t) \simeq \mu_0 n I_s(t)$, so the final answer can equally be written in terms of the source current I_s .

Use the slowly varying, long-solenoid approximation: ignore end effects and propagation across the apparatus. Rotational symmetry makes the azimuthal component E_ϕ constant on a circle of radius ϱ .



Violet arrows show the actual electric-field direction when $\dot{B} > 0$. For the calculation, take the positive direction to be counterclockwise, so $E_\phi < 0$.

Example — one flux rule, two radii

Find E_ϕ inside and outside the solenoid.

Solution. On a circle, $\oint \mathbf{E} \cdot d\mathbf{\ell} = 2\pi\varrho E_\phi$. The field-filled area is $\pi\varrho^2$ inside and πR^2 outside:

$$2\pi\varrho E_\phi = \begin{cases} -\pi\varrho^2 \dot{B}, & \varrho < R, \\ -\pi R^2 \dot{B}, & \varrho > R. \end{cases}$$

Dividing by the circumference gives

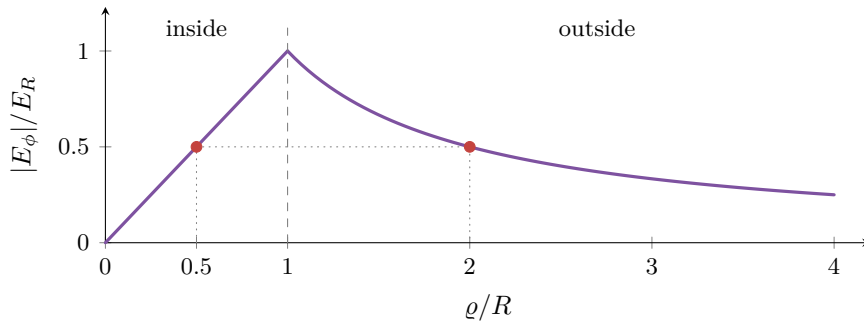
$$\mathbf{E} = E_\phi \hat{\phi}, \quad E_\phi(\varrho, t) = \begin{cases} -\frac{\varrho}{2} \dot{B}(t), & 0 \leq \varrho \leq R, \\ -\frac{R^2}{2\varrho} \dot{B}(t), & \varrho \geq R. \end{cases} \quad (3.1)$$

Physical meaning

The induced electric field extends outside the magnetic region. An exterior loop encloses changing flux even where the local magnetic field is negligible. No conducting ring is needed to create this electric field.

3.1 Read the result and check its curl

The electric field grows linearly with radius inside and falls as $1/\varrho$ outside. Its magnitude is largest at R and is continuous there. For $\dot{B} \neq 0$, define $E_R = R|\dot{B}|/2$.



The same radial shape appeared for A_ϕ in Week 6. Here its amplitude is set by the rate of change of the magnetic field. The marked points illustrate $\kappa = 1/2$ in the example below.

The integral law gives a local curl law

For a fixed surface, Stokes' theorem turns Faraday's law into

$$\int_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} \, dS = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} \, dS.$$

Since this holds for every small surface,

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (3.2)$$

For our azimuthal field, $(\nabla \times \mathbf{E})_z = \varrho^{-1} \partial_\varrho (\varrho E_\phi)$. Equation (3.1) gives $-\dot{B}$ inside and zero outside, as required.

Example — equal fields at unequal radii

Choose $0 < \kappa < 1$. Compare the field at the inner radius $\varrho_1 = \kappa R$ and outer radius $\varrho_2 = R/\kappa$.

Solution. Substitution into the two branches gives

$$E_\phi(\varrho_1) = E_\phi(\varrho_2) = -\frac{\kappa R}{2} \dot{B}.$$

The fields agree, but their circulations do not:

$$\mathcal{E}_1 = -\pi \kappa^2 R^2 \dot{B}, \quad \mathcal{E}_2 = -\pi R^2 \dot{B}.$$

The outer path is $1/\kappa^2$ times longer and encloses that much more changing flux.

Check / physical lesson

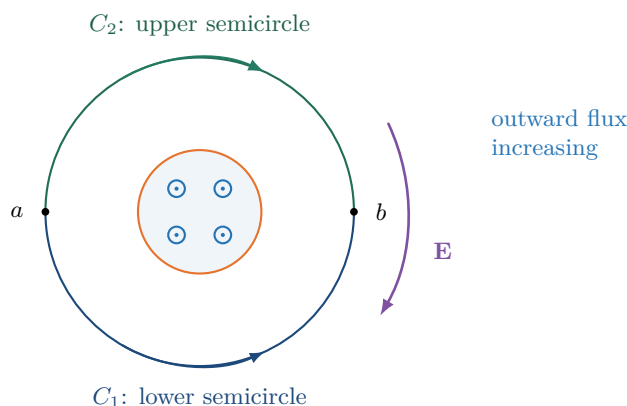
In the symmetric gauge of Week 6, $A_\phi = B\varrho/2$ inside and $A_\phi = BR^2/(2\varrho)$ outside. With a spatially constant scalar potential, $E_\phi = -\partial_t A_\phi$ reproduces both branches directly.

4 Path dependence: the endpoints are not enough

In electrostatics, $\int_a^b \mathbf{E} \cdot d\boldsymbol{\ell}$ depends only on the endpoints. We now compare two fixed paths around a changing flux tube and see what fails. Let both C_1 and C_2 run from a to b ; the closed path is $C_1 - C_2$. Then

$$\int_{C_1} \mathbf{E} \cdot d\boldsymbol{\ell} - \int_{C_2} \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}. \quad (4.1)$$

The minus sign on C_2 means that we follow it backwards when closing the loop.



Both path arrows go from a to b . The electric field is clockwise, so it aids travel along C_2 and opposes travel along C_1 .

Example — two routes and any number of extra turns

Use the two semicircles of radius $\varrho > R$ in the diagram. Let the linked outward flux be any smooth function $\Phi_B(t)$. Find the two line integrals.

Solution. Outside the solenoid, $E_\phi = -\dot{\Phi}_B/(2\pi\varrho)$. The lower path follows $+\hat{\phi}$; the upper path follows $-\hat{\phi}$. Hence

$$\int_{C_1} \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{\dot{\Phi}_B}{2}, \quad \int_{C_2} \mathbf{E} \cdot d\boldsymbol{\ell} = +\frac{\dot{\Phi}_B}{2}.$$

Their difference is $-\dot{\Phi}_B$, independent of the radius. Adding m complete counterclockwise turns to C_1 changes its integral to $-(m + \frac{1}{2})\dot{\Phi}_B$, with integer m .

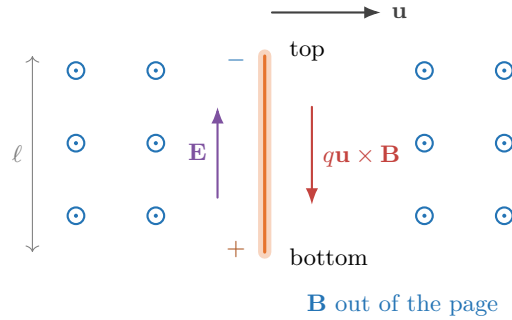
Physical meaning

A single scalar potential cannot describe this entire electric field as $-\nabla\Phi$. Outside the flux tube the local curl is zero, yet the circulation around the hole is nonzero. A scalar potential still exists as part of the full potential description, but its difference alone does not give these line integrals.

5 Motional emf: move the conductor, keep the field fixed

We now calculate an open-circuit voltage from the Lorentz force. Use the laboratory frame and nonrelativistic motion; neglect magnetic self-fields. Let a straight conducting rod of length ℓ move with velocity $\mathbf{u} = v\hat{\mathbf{x}}$ through a steady field $\mathbf{B} = B\hat{\mathbf{z}}$. The rod lies along y . Positive charge is pushed toward its lower end because

$$\frac{\mathbf{F}_B}{q} = \mathbf{u} \times \mathbf{B} = -vB\hat{\mathbf{y}}.$$



An open rod develops surface charges. Their upward electric field balances the downward magnetic force per positive charge.

The open-circuit voltage follows from force balance

After the brief charge-separation transient, no current flows along the rod. The tangential forces balance:

$$E_y + (\mathbf{u} \times \mathbf{B})_y = 0 \implies E_y = vB.$$

Integrating from bottom to top gives $V_{\text{top}} - V_{\text{bottom}} = -\int_0^\ell vB \, dy = -vB\ell$. Hence

$$\boxed{V_{\text{bottom}} - V_{\text{top}} = B\ell v.} \quad (5.1)$$

Example — the rod need not be perpendicular to its motion

Keep $\mathbf{u} = v\hat{\mathbf{x}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$, but tilt the rod so that $\mathbf{r}_2 - \mathbf{r}_1 = \ell(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}})$. Find $V_1 - V_2$ after charge separation.

Solution. Along the rod, equilibrium requires $\mathbf{E} \cdot d\boldsymbol{\ell} = -(\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}$. Hence

$$V_2 - V_1 = (\mathbf{u} \times \mathbf{B}) \cdot (\mathbf{r}_2 - \mathbf{r}_1) = -vB\ell \cos\theta.$$

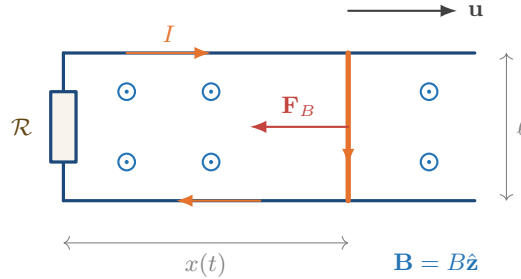
Thus $\boxed{V_1 - V_2 = vB\ell \cos\theta}$. Only the rod's projection along y matters. The vertical rod in the diagram has $\theta = 0$; a rod parallel to its motion has zero terminal voltage.

Check / physical lesson

The applied magnetic field is steady in the laboratory: $\partial_t \mathbf{B} = 0$. The driving term here is $\mathbf{u} \times \mathbf{B}$. In the solenoid example the loop was fixed and the driving term was the induced $\mathbf{E}$.

6 Close the circuit: current, drag, and power

Place the moving rod on conducting rails joined by a resistor $\mathcal{R}$. We now calculate the current and the force needed to maintain its speed. Neglect the rod and rail resistances, self-inductance, and the induced field compared with the applied field.



The outward flux grows as the rod moves right. The induced current is clockwise; its force on the moving rod points left.

Choose $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and positive circulation counterclockwise. The area is $S = \ell x$, so $\Phi_B = B\ell x$ and $\mathcal{E} = -B\ell v$. The clockwise current has magnitude

$$I = \frac{B\ell v}{\mathcal{R}}.$$

On the rod the current points downward. Week 5's wire-force rule gives

$$\mathbf{F}_B = -I\ell B\hat{\mathbf{x}} = -\frac{B^2\ell^2 v}{\mathcal{R}}\hat{\mathbf{x}}. \quad (6.1)$$

Example — release the rod: induction gives magnetic braking

Let the rod have mass m and initial speed $v_0 > 0$. Remove the external pull and neglect friction. Find $v(t)$ and the total heat produced.

Solution. Newton's law and Eq. (6.1) give

$$m\dot{v} = -\frac{B^2\ell^2}{\mathcal{R}}v, \quad \boxed{v(t) = v_0 e^{-t/\tau}, \quad \tau = \frac{m\mathcal{R}}{B^2\ell^2}}.$$

The current magnitude is $I(t) = B\ell v(t)/\mathcal{R}$. Therefore

$$\int_0^\infty I^2 \mathcal{R} dt = \frac{B^2\ell^2 v_0^2}{\mathcal{R}} \frac{\tau}{2} = \boxed{\frac{1}{2}mv_0^2}.$$

All of the initial kinetic energy becomes heat in this ideal model.

Physical meaning

The magnetic force on an individual charge still does no work: $q(\mathbf{v}_{\text{charge}} \times \mathbf{B}) \cdot \mathbf{v}_{\text{charge}} = 0$. When an external pull maintains v , its power is $P_{\text{pull}} = |\mathbf{F}_B|v = I^2\mathcal{R}$. After release, the rod supplies that energy from its own motion. Week 8 describes the same transfer through the electromagnetic fields.

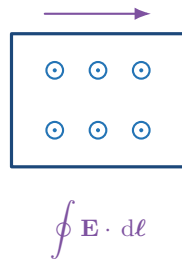
7 A moving loop: combine the two contributions

For a thin moving circuit, let $\mathbf{u}$ be the local velocity of the wire. The electromagnetic emf at an instant is

$$\mathcal{E} = \oint_{C(t)} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot \hat{\mathbf{n}} dS. \quad (7.1)$$

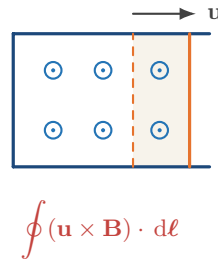
The total derivative counts both a field that changes at fixed positions and a boundary that moves or deforms. The local Maxwell law still uses $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$. The moving-surface derivation is given in Sec. 17.

fixed loop, growing field



$$\oint \mathbf{E} \cdot d\boldsymbol{\ell}$$

moving boundary, fixed field



$$\oint (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}$$

The two effects can occur separately or together. In each panel, the direction follows from the chosen flux normal and the Lorentz force.

Example — two nonzero contributions can cancel

A rectangular loop has height ℓ and width $x(t)$ in $\mathbf{B} = B(t)\hat{\mathbf{z}}$. Let $v = \dot{x}$. Find its emf, then find a motion with zero emf.

Solution. Differentiate the entire linked flux:

$$\Phi_B = \ell x B \quad \Longrightarrow \quad \mathcal{E} = \underbrace{-\ell x \dot{B}}_{\text{changing field}} \quad \underbrace{-\ell B \dot{x}}_{\text{motion}}.$$

The two terms cancel if $x\dot{B} + B\dot{x} = 0$, or

$$x(t)B(t) = x_0 B_0, \quad x(t) = \frac{x_0 B_0}{B(t)}$$

for $B(t) \neq 0$, with $x_0 = x(0)$ and $B_0 = B(0)$. A shrinking loop in a rising field can therefore have zero emf. Neither the induced electric field nor the magnetic force needs to vanish.

Check / physical lesson

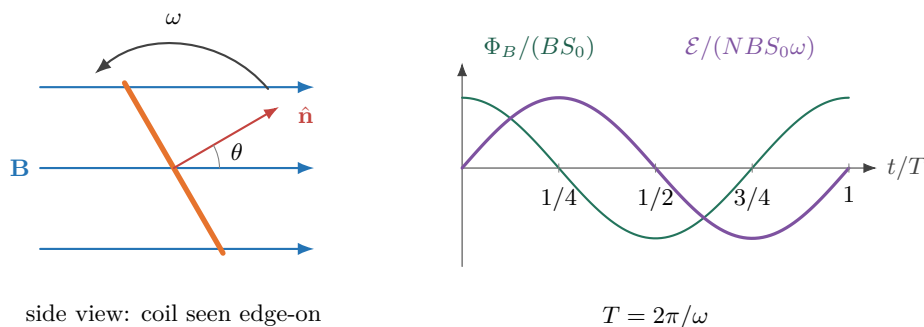
A changing *flux* does not always mean a changing local $\mathbf{B}$. Area or orientation can change instead. The flux rule applies to a well-defined evolving circuit path; it is not a rule for an arbitrary moving area unrelated to the conducting path.

8 A rotating coil: build an alternating emf

A generator changes flux by changing orientation. Let a coil of N identical turns, each of area S_0 , rotate at angular speed ω in a uniform steady field. The angle between its normal and the field is $\theta = \omega t$. The flux per turn is $\Phi_B = BS_0 \cos \theta$; the total emf is

$$\mathcal{E} = -N \frac{d\Phi_B}{dt} = NBS_0\omega \sin(\omega t). \quad (8.1)$$

The factor N counts the same flux linked by each turn connected in series.



The normal is perpendicular to the orange coil plane. Flux and emf are shifted by a quarter cycle: the emf is largest when the flux crosses zero.

Example — derive the load power and the resisting torque

Connect the coil to a total resistance $\mathcal{R}$ and neglect self-inductance. Find the current, average power, and torque required to maintain $\omega > 0$.

Solution. With $\mathcal{E}_0 = NBS_0\omega$,

$$I(t) = \frac{\mathcal{E}_0}{\mathcal{R}} \sin(\omega t), \quad P(t) = I^2 \mathcal{R} = \frac{\mathcal{E}_0^2}{\mathcal{R}} \sin^2(\omega t).$$

Because the average of $\sin^2$ over one turn is $1/2$,

$$\langle P \rangle = \frac{(NBS_0\omega)^2}{2\mathcal{R}}.$$

The magnetic torque along the rotation axis is $\tau_B = -NIS_0B \sin \theta$. Hence the external torque obeys $\tau_{\text{ext}}\omega = \mathcal{E}I = P(t)$.

Physical meaning

Emf depends on the *rate of change* of flux, not on the amount of flux alone. Doubling N , B , or S_0 doubles the emf and quadruples the power at fixed resistance and speed. The required mechanical power grows by the same factor.

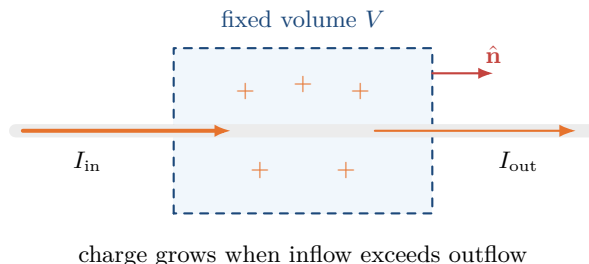
We have coupled magnetic change to electric circulation. Charge conservation will show why the reverse coupling is needed.

9 Charge conservation: allow charge to accumulate

Week 5 used steady currents, for which charge did not build up anywhere. We now keep the time dependence. For a fixed volume V , let $Q_V = \int_V \rho \, dV$, where ρ is charge density and $\mathbf{J}$ is current density. Positive outward current removes positive charge from the volume:

$$\frac{dQ_V}{dt} = - \oint_{\partial V} \mathbf{J} \cdot \hat{\mathbf{n}} \, dS. \quad (9.1)$$

Here $\hat{\mathbf{n}}$ points outward on the closed boundary ∂V .



Current through a boundary and charge inside a volume are different quantities. Their difference determines the rate of accumulation.

Turn the bookkeeping statement into a local equation

For fixed V , move the time derivative inside the volume integral and use the divergence theorem:

$$\int_V \frac{\partial \rho}{\partial t} \, dV = - \int_V \nabla \cdot \mathbf{J} \, dV.$$

Since the volume can be chosen arbitrarily small,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (9.2)$$

This is the continuity equation. Positive divergence means net outward flow and decreasing charge density.

Example — a current pulse builds a finite plate charge

A capacitor plate initially has charge Q_0 . Current $I(t) = I_0 e^{-t/\tau}$ enters it for $t \geq 0$, with no outgoing conduction current. Find the plate charge.

Solution. Apply the integral continuity equation to a volume enclosing the plate:

$$\dot{Q} = I(t), \quad Q(t) = Q_0 + \int_0^t I(t') \, dt' = Q_0 + I_0 \tau (1 - e^{-t/\tau}).$$

Initially $\dot{Q} = I_0$; eventually the added charge is $I_0 \tau$. No charge needs to cross the vacuum gap: it accumulates on the plate.

10 Maxwell's correction: make the current term consistent

We now ask whether the magnetostatic law can describe a region where charge accumulates. Taking the divergence of $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ gives

$$0 = \mu_0 \nabla \cdot \mathbf{J}.$$

But continuity requires $\nabla \cdot \mathbf{J} = -\partial_t \rho$. The magnetostatic law therefore fails when ρ changes.

Use Gauss's law to find a consistent source term

From $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$ and charge conservation,

$$\nabla \cdot \mathbf{J} = -\varepsilon_0 \partial_t (\nabla \cdot \mathbf{E}), \quad \nabla \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = 0.$$

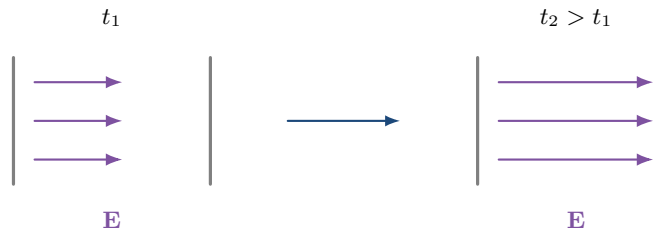
The combination in parentheses has zero divergence, as the right-hand side of a curl equation must. Maxwell's full law is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (10.1)$$

Define the **displacement-current density** in vacuum by

$$\mathbf{J}_D = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad [J_D] = \text{A/m}^2. \quad (10.2)$$

It contributes to magnetic circulation even in a region where $\mathbf{J} = 0$.



growing rightward electric field: $\mathbf{J}_D$ points right

The arrows indicate field strength, not particles crossing the gap. Displacement current is a field term, not ordinary transport of charge through vacuum.

Example — field amplitude and time scale play different roles

A uniform electric field is $\mathbf{E} = E_0 f(t/\tau) \hat{\mathbf{z}}$, where f is dimensionless and a prime denotes its derivative. Find the displacement current through a perpendicular area S_0 , with normal $+\hat{\mathbf{z}}$.

Solution. Differentiate at a fixed position:

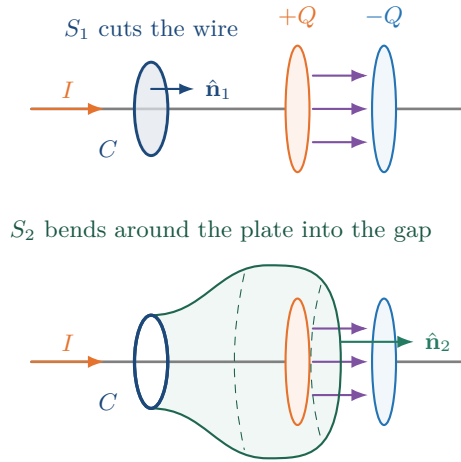
$$\mathbf{J}_D = \frac{\varepsilon_0 E_0}{\tau} f'(t/\tau) \hat{\mathbf{z}}, \quad I_D = \frac{\varepsilon_0 S_0 E_0}{\tau} f'(t/\tau).$$

At fixed field profile, doubling the amplitude doubles I_D ; stretching the time scale by two halves it. A constant value of f gives $I_D = 0$.

Charge conservation motivates this correction; it does not uniquely determine every possible term in a dynamical law. The Ampère–Maxwell equation is the experimentally tested extension. A time-dependent current can itself have zero divergence, but that does not justify discarding the displacement term in general.

11 A charging capacitor: one loop, two spanning surfaces

A current I charges a capacitor, but no conduction current crosses its vacuum gap. We now compare two surfaces bounded by the *same* Ampèrian loop C . This makes the correction visible geometrically.



The blue boundary C stays fixed. The green surface avoids the wire and passes between the plates. Surfaces are mathematical integration surfaces, not physical membranes.

By Stokes' theorem, Eq. (10.1) becomes

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 (I_{\text{cond}} + I_D), \quad I_D = \varepsilon_0 \frac{d\Phi_E}{dt}. \quad (11.1)$$

Here $I_{\text{cond}} = \int_S \mathbf{J} \cdot \hat{\mathbf{n}} dS$ and $\Phi_E = \int_S \mathbf{E} \cdot \hat{\mathbf{n}} dS$ refer to the *chosen fixed surface*.

Example — the equality holds for any charging history

Use ideal plates of area S_p and separation d , with charge $\pm Q(t)$. Show that either surface gives the same magnetic circulation.

Solution. In the gap, $E = Q/(\varepsilon_0 S_p)$, so

$$I_D = \varepsilon_0 \frac{d(ES_p)}{dt} = \dot{Q} = I.$$

The wire-cutting surface counts $I_{\text{cond}} = I$; the gap surface counts $I_D = I$. Both give $\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I$. Equivalently, with $\Delta V = Ed$ and $C = \varepsilon_0 S_p/d$,

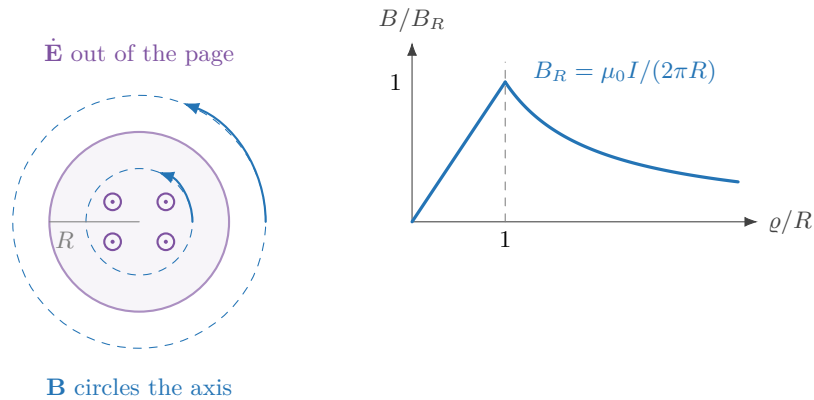
$$Q = C\Delta V, \quad I = I_D = C \frac{d\Delta V}{dt}.$$

12 Inside the capacitor gap: calculate the magnetic field

Take circular plates of radius R , separated by a small gap $d \ll R$. The charging current is $I(t) = \dot{Q}(t)$. Use the slowly varying parallel-plate approximation: ignore fringing and propagation across the capacitor. Then

$$E(t) = \frac{Q(t)}{\varepsilon_0 \pi R^2}, \quad \dot{E}(t) = \frac{I(t)}{\varepsilon_0 \pi R^2}.$$

Choose $+z$ from the positive plate to the negative plate, so $\mathbf{E} = E\hat{\mathbf{z}}$ when $Q > 0$. We seek the azimuthal $\mathbf{B}$ on a circle of radius ϱ centred on the axis, away from the return wire.



Looking along the capacitor axis, the growing electric field plays the role of an axial current density. The blue arrows are for $I > 0$.

Example — count only the enclosed displacement current

Find B_ϕ inside and outside the plate radius, within this ideal field model.

Solution. In the gap, $I_{\text{cond}} = 0$. For $\varrho < R$, $I_D = \varepsilon_0 \dot{E} \pi \varrho^2 = I \varrho^2 / R^2$. For $\varrho > R$, the full electric flux is enclosed, so $I_D = I$. Using $2\pi \varrho B_\phi = \mu_0 I_D$ gives

$$B_\phi(\varrho, t) = \begin{cases} \frac{\mu_0 I(t) \varrho}{2\pi R^2}, & 0 \leq \varrho \leq R, \\ \frac{\mu_0 I(t)}{2\pi \varrho}, & \varrho \geq R. \end{cases} \quad (12.1)$$

At the rim, $B_R = \mu_0 I / (2\pi R)$. For $0 < \kappa < 1$, both $\varrho = \kappa R$ and R/κ give $B_\phi = \kappa B_R$: the same two-branch geometry as the induced solenoid field.

Check / physical lesson

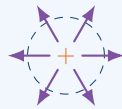
The field is finite at the axis and continuous at R . Its radial form is the same as the uniformly current-filled wire in Week 5, with $\mathbf{J}$ replaced by $\mathbf{J}_D$. The exterior branch applies away from the return wire in this near-field model. The charging time scale must be long compared with R/c , where c is the speed of light; rapid charging requires spatial and propagation corrections.

For Week 8. At the rim, we now know both $E = Q/(\varepsilon_0 \pi R^2)$ and $B_R = \mu_0 I / (2\pi R)$. The first measures stored charge, the second its rate of change. These are the fields needed to calculate energy entering the capacitor.

13 Maxwell's equations: two flux laws, two circulation laws

We now collect the equations that determine the fields and constrain their sources. These are the vacuum equations *with sources* ρ and $\mathbf{J}$; “vacuum” here means that no material polarization or magnetization is included.

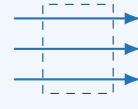
Electric flux counts charge



$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\oint_{\partial V} \mathbf{E} \cdot \hat{\mathbf{n}} dS = \frac{Q_V}{\epsilon_0}$$

Magnetic flux has no source



$$\nabla \cdot \mathbf{B} = 0$$

$$\oint_{\partial V} \mathbf{B} \cdot \hat{\mathbf{n}} dS = 0$$

Changing magnetic flux



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}$$

Current and changing electric flux



$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 (I_{\text{cond}} + I_D)$$

Here $\Phi_E = \int_S \mathbf{E} \cdot \hat{\mathbf{n}} dS$, $\Phi_B = \int_S \mathbf{B} \cdot \hat{\mathbf{n}} dS$, and $I_D = \epsilon_0 \dot{\Phi}_E$. For both circulation laws, $C = \partial S$ and S is fixed in space. For both flux laws, ∂V is a closed surface with outward normal.

Example — a large field is not necessarily a displacement current

A disconnected capacitor holds a constant electric field in its gap. Does that field alone supply displacement current?

Solution. No. $\mathbf{J}_D = \epsilon_0 \partial_t \mathbf{E} = 0$ because the field is constant in time. The electric flux can be nonzero while its time derivative is zero.

Physical meaning

Divergence laws concern flux through a *closed* surface. Curl laws concern circulation around the boundary of an *open* surface. Choosing the correct geometry is as important as choosing the correct equation.

14 Time-dependent potentials: keep both contributions

We now complete the potential description begun in Week 6. Write $\mathbf{B} = \nabla \times \mathbf{A}$ in Faraday's law. Then

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

In a simply connected region (one without holes that obstruct contracting a loop), the curl-free combination is a gradient. Thus

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}.} \quad (14.1)$$

These expressions automatically satisfy magnetic Gauss law and Faraday's law. A time-dependent gauge function $\chi(\mathbf{r}, t)$ changes *both* potentials:

$$\boxed{\mathbf{A}' = \mathbf{A} + \nabla \chi, \quad \Phi' = \Phi - \frac{\partial \chi}{\partial t}.} \quad (14.2)$$

The curl of a gradient is zero. In $\mathbf{E}'$, the two new mixed derivatives cancel because $\nabla \partial_t \chi = \partial_t \nabla \chi$. Both physical fields are unchanged.

Example — a changing vector potential with zero physical fields

Start from $\mathbf{A} = 0$ and $\Phi = 0$. Choose $\chi = f(t)x$, where $f(t)$ is any smooth function. What fields follow from the transformed potentials?

Solution. The new potentials are

$$\mathbf{A}' = f(t)\hat{\mathbf{x}}, \quad \Phi' = -\dot{f}(t)x.$$

Their derivatives give

$$\mathbf{B}' = \nabla \times \mathbf{A}' = 0, \quad \mathbf{E}' = \underbrace{+\dot{f}\hat{\mathbf{x}}}_{-\nabla \Phi'} + \underbrace{-\dot{f}\hat{\mathbf{x}}}_{-\partial_t \mathbf{A}'} = 0.$$

A time-dependent $\mathbf{A}$ alone does not guarantee a nonzero electric field.

The split into scalar- and vector-potential contributions depends on gauge. Only their sum is the electric field. Arrows are shown for $\dot{f} > 0$.

Check / physical lesson

For a fixed path from a to b , the potential formula also gives

$$\int_a^b \mathbf{E} \cdot d\boldsymbol{\ell} = \Phi(a) - \Phi(b) - \frac{\partial}{\partial t} \int_a^b \mathbf{A} \cdot d\boldsymbol{\ell}.$$

The scalar potential Φ differs from magnetic flux Φ_B . Only the full combination above is gauge invariant; a changing-field example appears in Sec. 18.

15 Fields can propagate: the first view of a wave

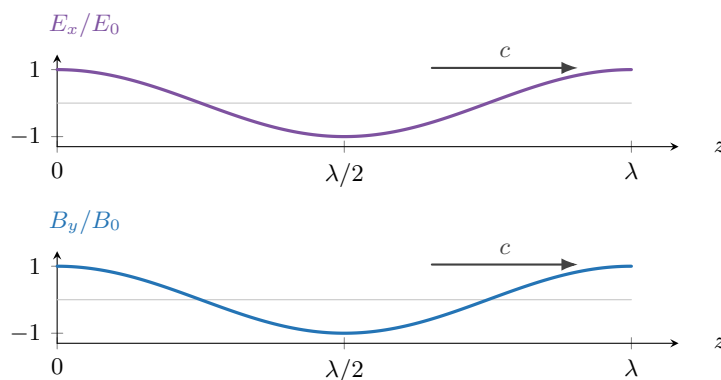
We finish by asking what Maxwell's equations allow where there is *no local charge or current*. The two curl laws still couple $\mathbf{E}$ and $\mathbf{B}$. They admit travelling fields with speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \simeq 3.00 \times 10^8 \text{ m s}^{-1}. \quad (15.1)$$

One example travelling in the $+z$ direction is

$$\mathbf{E} = E_0 \cos(kz - \omega t) \hat{\mathbf{x}}, \quad \mathbf{B} = \frac{E_0}{c} \cos(kz - \omega t) \hat{\mathbf{y}}, \quad \omega = ck. \quad (15.2)$$

Here $k = 2\pi/\lambda$ is the wave number and $\omega = 2\pi f$ is the angular frequency. The fields are perpendicular, peak together, and are transverse: neither points along z . We verify their amplitude ratio and speed directly below.



Two component plots at the same instant, with $B_0 = E_0/c$. The curves show field values along z , not the paths followed by charges.

Example — derive the speed and amplitude ratio together

Try $E_x = E_0 \cos(kz - \omega t)$ and $B_y = B_0 \cos(kz - \omega t)$, with $k, \omega > 0$. What do the two curl laws require?

Solution. Their nonzero components are $\partial_z E_x = -\partial_t B_y$ and $-\partial_z B_y = c^{-2} \partial_t E_x$. Substitution gives

$$kE_0 = \omega B_0, \quad kB_0 = \frac{\omega}{c^2} E_0 \quad \implies \quad \boxed{\omega = ck, \quad B_0 = E_0/c.}$$

A crest has constant $kz - \omega t$, so it moves at $\dot{z} = \omega/k = c$. The cross product $\mathbf{E} \times \mathbf{B}$ points along $+z$, even when both fields reverse.

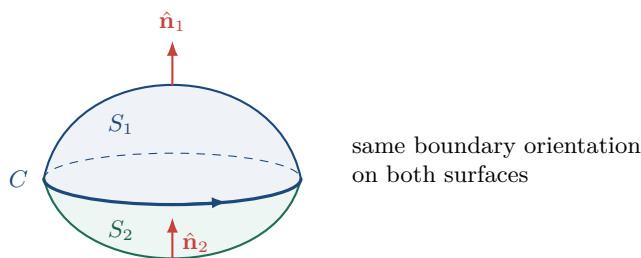
Bridge to Week 8. The moving rod established a power balance; the capacitor supplied a matched pair of fields; the potentials separated a representation from measurable fields. Week 8 uses these results to derive field energy flow, momentum, and forces. Gauge conditions organize the same dynamics.

OPTIONAL — Why the spanning surface does not change the answer

Further detail, following the main development.

16 One boundary, two surfaces

We now justify a geometric fact used in both circulation laws: changing the spanning surface must not change the answer around its fixed boundary. Let S_1 and S_2 share C , with normals chosen to give the same positive direction around C .



The upper and lower caps meet along the same rim. To form an outward-oriented *closed* surface, use S_1 and $-S_2$: reverse the lower normal.

Derivation — magnetic flux

The divergence theorem applied to the volume between the surfaces gives

$$\int_{S_1} \mathbf{B} \cdot \hat{n}_1 dS - \int_{S_2} \mathbf{B} \cdot \hat{n}_2 dS = \int_V \nabla \cdot \mathbf{B} dV = 0.$$

Thus $\Phi_B[S_1] = \Phi_B[S_2]$. This assumes the field is defined throughout the intervening volume and that magnetic Gauss law holds there.

Derivation — total current

Gauss's electric law and charge conservation give

$$\nabla \cdot \mathbf{J}_{\text{tot}} = \nabla \cdot \mathbf{J} + \varepsilon_0 \partial_t (\nabla \cdot \mathbf{E}) = \nabla \cdot \mathbf{J} + \partial_t \rho = 0, \quad \mathbf{J}_{\text{tot}} = \mathbf{J} + \varepsilon_0 \partial_t \mathbf{E}.$$

The same volume argument therefore gives

$$\int_{S_1} \mathbf{J}_{\text{tot}} \cdot \hat{n}_1 dS = \int_{S_2} \mathbf{J}_{\text{tot}} \cdot \hat{n}_2 dS.$$

Conduction current alone need not agree on the two surfaces; the *sum* with displacement current does.

Example — a loop outside a changing solenoid

Outside the ideal long solenoid, $\partial_t \mathbf{B} = 0$ and hence $\nabla \times \mathbf{E} = 0$ locally. Can a loop surrounding the solenoid nevertheless have nonzero circulation?

Solution. Yes. Its circulation is $-\pi R^2 \dot{B}$. Every spanning surface for a loop linked once with the ideal infinite solenoid crosses its field region. There is no spanning surface lying entirely in the exterior on which to apply the local zero-curl statement.

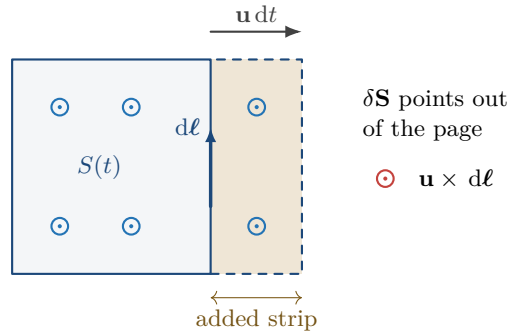
Physical meaning

A local equation and a global integral agree only when the chosen surface lies in a region where the equation and fields apply. Geometry matters even when the local curl is zero.

OPTIONAL — A short geometric proof of the moving-loop flux rule
Further detail, following the main development.

17 The swept strip: why the motional term appears

We now calculate how magnetic flux changes when the boundary moves. There are two changes: the field changes on the old surface, and the moving boundary sweeps out a strip. The instantaneous surface normal and positive loop direction obey the right-hand rule.



For the right edge of a counterclockwise loop, $d\ell$ points up. Motion to the right adds an outward-oriented strip. The field symbols show $\mathbf{B}$ out of the page.

Derivation — follow the changing area

To first order in dt , the strip has oriented area

$$\delta\mathbf{S} = (\mathbf{u} \times d\ell) dt.$$

Its added flux is $\mathbf{B} \cdot \delta\mathbf{S} = -(\mathbf{u} \times \mathbf{B}) \cdot d\ell dt$. Adding this contribution around the moving boundary gives

$$\frac{d\Phi_B}{dt} = \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} dS - \oint_{C(t)} (\mathbf{u} \times \mathbf{B}) \cdot d\ell. \quad (17.1)$$

Magnetic Gauss law ensures that an interior deformation with the same boundary adds no extra term. At each instant, local Faraday law and Stokes' theorem give

$$\oint_{C(t)} \mathbf{E} \cdot d\ell = - \int_{S(t)} \partial_t \mathbf{B} \cdot \hat{\mathbf{n}} dS.$$

Combining the two results,

$$\oint_{C(t)} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot d\ell = - \frac{d\Phi_B}{dt}.$$

Example — a whole loop translating through a uniform field

A rigid closed loop translates with constant velocity $\mathbf{u}$ through a uniform static $\mathbf{B}$. Does the magnetic force produce a net emf?

Solution. No. Its area and orientation stay fixed, so its flux stays fixed. Directly, the motional contribution is

$$\oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\ell = (\mathbf{u} \times \mathbf{B}) \cdot \underbrace{\oint_C d\ell}_0 = 0.$$

Opposite parts of the loop cancel. Local magnetic forces do not by themselves guarantee a net circuit emf.

OPTIONAL — The same changing field in two different gauges
 Further detail, following the main development.

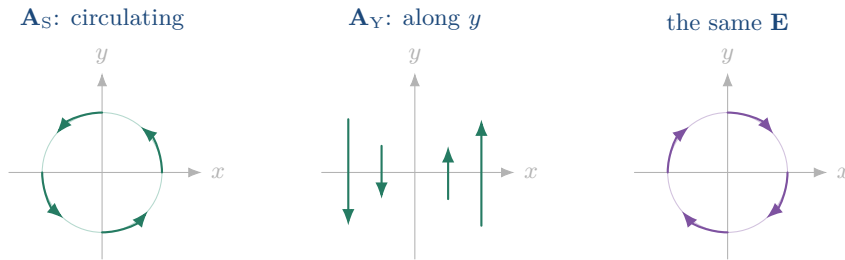
18 A concrete gauge change

Take a uniform field $\mathbf{B} = B(t)\hat{\mathbf{z}}$ in an interior region, with $B(t) = B_0 + \alpha t$ and constant ramp rate α . This calculation shows why changing $\mathbf{A}$ requires a matching change of Φ . Start with the symmetric choice

$$\mathbf{A}_S = \frac{B(t)}{2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}), \quad \Phi_S = 0.$$

It gives the circulating electric field

$$\mathbf{E} = -\partial_t \mathbf{A}_S = \frac{\alpha}{2}(y\hat{\mathbf{x}} - x\hat{\mathbf{y}}). \quad (18.1)$$



Two different vector potentials, one physical electric field. Arrows are shown for $B > 0$ and $\alpha > 0$; the diagrams display directions, not a common magnitude scale.

Example — transform both potentials and recover the field

Choose $\chi = B(t)xy/2$. Find the transformed potentials and electric field.

Solution. Since $\nabla\chi = \frac{B}{2}(y\hat{\mathbf{x}} + x\hat{\mathbf{y}})$ and $\partial_t\chi = \alpha xy/2$,

$$\mathbf{A}_Y = Bx\hat{\mathbf{y}}, \quad \Phi_Y = -\frac{\alpha}{2}xy.$$

The two electric-field contributions are now

$$-\nabla\Phi_Y = \frac{\alpha}{2}(y\hat{\mathbf{x}} + x\hat{\mathbf{y}}), \quad -\partial_t\mathbf{A}_Y = -\alpha x\hat{\mathbf{y}}.$$

Adding them gives

$$\mathbf{E} = \frac{\alpha}{2}(y\hat{\mathbf{x}} - x\hat{\mathbf{y}}),$$

exactly as in Eq. (18.1). Keeping only $-\partial_t\mathbf{A}_Y$ would miss part of the physical field.

Check / physical lesson

Check the fields directly:

$$(\nabla \times \mathbf{A}_Y)_z = \partial_x(Bx) = B, \quad (\nabla \times \mathbf{E})_z = \partial_x\left(-\frac{\alpha x}{2}\right) - \partial_y\left(\frac{\alpha y}{2}\right) = -\alpha.$$

The potentials look different, but both magnetic Gauss law and Faraday's law agree. The labels S and Y mean symmetric and y -directed. Both have $\nabla \cdot \mathbf{A} = 0$, the Coulomb condition studied in Week 8. For this linear ramp, $\partial_t \mathbf{E} = 0$ and $\nabla \times \mathbf{B} = 0$, so the vacuum Ampère–Maxwell equation also holds in the region.

OPTIONAL — How Maxwell's equations evolve consistent fields
Further detail, following the main development.

19 Constraints and evolution

We close with two short calculations: why the divergence laws remain valid as fields evolve, and how the curl laws give the wave speed used on the last main page.

1. The divergence laws constrain the initial fields. The curl laws can be written as evolution equations:

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad \partial_t \mathbf{E} = c^2 \nabla \times \mathbf{B} - \frac{\mathbf{J}}{\varepsilon_0}, \quad c^2 = \frac{1}{\mu_0 \varepsilon_0}.$$

Taking their divergences gives

$$\begin{aligned} \partial_t (\nabla \cdot \mathbf{B}) &= 0, \\ \partial_t \left(\nabla \cdot \mathbf{E} - \frac{\rho}{\varepsilon_0} \right) &= -\frac{\nabla \cdot \mathbf{J} + \partial_t \rho}{\varepsilon_0} = 0. \end{aligned}$$

The second equality uses charge conservation. If both Gauss laws hold initially, the evolution preserves them. The curl laws alone do not correct initial fields that violate these constraints.

2. In a source-free region, the curl laws give a wave equation. Take $\rho = 0$, $\mathbf{J} = 0$, and fields of the form $\mathbf{E} = E_x(z, t)\hat{\mathbf{x}}$, $\mathbf{B} = B_y(z, t)\hat{\mathbf{y}}$. Their divergences vanish automatically. The nonzero curl components give

$$\partial_z E_x = -\partial_t B_y, \quad -\partial_z B_y = \frac{1}{c^2} \partial_t E_x.$$

Differentiate the first equation with respect to z and use the second:

$$\partial_z^2 E_x = -\partial_t (\partial_z B_y) = \frac{1}{c^2} \partial_t^2 E_x.$$

Thus

$$\boxed{\partial_z^2 E_x - \frac{1}{c^2} \partial_t^2 E_x = 0.} \quad (19.1)$$

The same steps give the corresponding wave equation for B_y .

Example — a general pulse, and why opposite directions matter

Let F and G be any smooth electric-field profiles. Show that

$$E_x = F(z - ct) + G(z + ct), \quad B_y = \frac{F(z - ct) - G(z + ct)}{c}$$

solves the two curl equations.

Solution. A prime denotes the derivative of each profile's argument. Then $\partial_z E_x = F' + G' = -\partial_t B_y$ and $-\partial_z B_y = (-F' + G')/c = c^{-2} \partial_t E_x$. The F pulse moves right; the G pulse moves left. Their magnetic fields have opposite signs for the same electric direction. For equal local values $F = G \neq 0$, $B_y = 0$ while $E_x = 2F$. Thus $|E_x| = c|B_y|$ holds for a single travelling wave, not every superposition.

Physical meaning

Waves, slowly varying induction fields, and static fields all obey the same Maxwell equations. The geometry and time scale decide which approximations are useful. In Week 8, the same distinction explains why stored field energy need not all flow in one direction.