Electrodynamics & Relativity

Week 10: Relativistic Electrodynamics

One interaction, two observers

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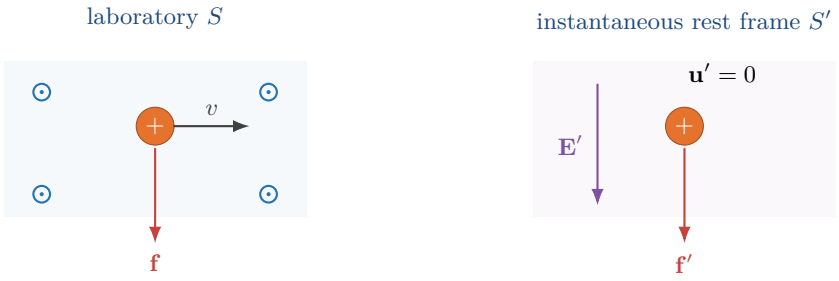
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The question for this week. How can two observers describe the same electromagnetic interaction when they measure different electric fields, magnetic fields, and particle velocities?
We will connect sources to potentials, potentials to fields, and fields to forces. The main route ends in Sec. 16. Four optional sections provide further applications and details; the core can be read without them.

1 One interaction: continue the moving-charge experiment

Week 9 ended with a charge moving through a magnetic field. We now calculate how that experiment is described by a second observer, using the same Lorentz transformation for every part of the description.



The same charge at the same event. The primed magnetic field is not drawn: it exerts no force on a charge whose velocity is zero at that instant.

Example — identify what the new description must explain

Take $q > 0$, $B_0 > 0$, and instantaneous laboratory velocity $v\hat{\mathbf{x}}$. What is the force, and what must the comoving observer find?

Solution. The laboratory Lorentz force is

$$\mathbf{f} = q\mathbf{u} \times \mathbf{B} = -qvB_0\hat{\mathbf{y}}.$$

In S' the magnetic term is zero at this event, because $\mathbf{u}' = 0$. An electric field must supply the force there. We will find $\mathbf{E}'$ in Sec. 9 and compare the two force readings in Sec. 13. The force magnitudes need not be equal.

Key relations

Throughout, S' moves at $+v\hat{\mathbf{x}}$ relative to S, with $|v| < c$. The origins coincide at $t = t' = 0$:

$$x' = \gamma(x - vt), \qquad t' = \gamma(t - vx/c^2), \qquad \beta = v/c, \quad \gamma = (1 - \beta^2)^{-1/2}.$$

We use SI units. A prime changes the observer; a bar will denote a gauge change. The symbol $\mathbf{u}$ is particle velocity, while v is frame velocity.

Next step. Put quantities that transform together into four-component objects.

2 Four-vectors: four numbers with one transformation rule

We need a way to recognize which measurements belong together. An event has four coordinates $x^\mu = (ct, x, y, z)$; multiplying time by c puts all four in units of length. A **four-vector** is a set of four components that transform in the same way as an event displacement.

$$V'^0 = \gamma(V^0 - \beta V^1), \qquad V'^1 = \gamma(V^1 - \beta V^0), \qquad V'^2 = V^2, \quad V'^3 = V^3.$$

Equivalently $V'^\mu = \Lambda^\mu{}_\nu V^\nu$, where $0, 1, 2, 3$ refer respectively to ct, x, y, z, and

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \eta = \mathrm{diag}(1, -1, -1, -1).$$

Method / strategy

An index is a component label, not a power. Greek indices run from 0 to 3; spatial indices i, j run from 1 to 3. A label repeated once up and once down is summed. A label left over is *free*: the equation says one thing for each of its values. In ordinary three-vector notation, subscripts such as E_y simply label Cartesian components.

The **metric** η records the signs of the interval. It lowers an index: $V_\mu = \eta_{\mu\nu} V^\nu = (V^0, -\mathbf{V})$. Thus $V_\mu W^\mu = V^0 W^0 - \mathbf{V} \cdot \mathbf{W}$ has the same value in both frames. Here $\mathbf{V} = (V^1, V^2, V^3)$ and W is another four-vector. The notation diag lists the diagonal entries; all other entries are zero. A superscript T means matrix transpose (exchange rows and columns).

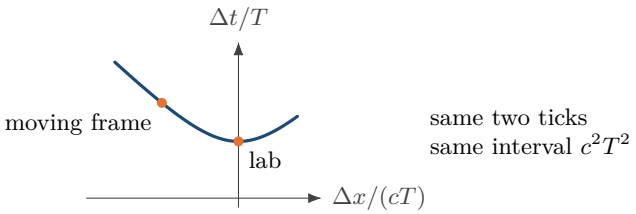
Example — two ticks of a clock at rest in the lab

A fixed lab clock ticks twice, T apart. Transform the separation and verify that both observers obtain the same interval.

Solution. The lab separation is $\Delta x^\mu = (cT, 0, 0, 0)$. Multiplying by Λ gives $(\gamma cT, -\gamma vT, 0, 0)$: the lab clock moves left in S'. The primed scalar product is

$$\Delta x'_\mu \Delta x'^\mu = \gamma^2(c^2 - v^2)T^2 = c^2 T^2.$$

The primed time component is larger; the spatial component is what restores the invariant result.



Plot the two pairs of measured components on common numerical axes. They lie on one hyperbola, because the difference of squares is fixed.

Next step. Use the proper time from Week 9 to construct the velocity and momentum four-vectors.

3 Velocity and momentum: use the particle's own clock

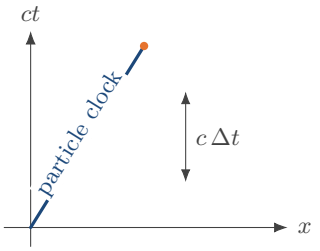
We now construct two four-vectors for a massive particle. They will let us describe its motion and the force acting on it in any inertial frame. The proper time τ is the reading of a clock travelling with that particle. For ordinary velocity $\mathbf{u} = \mathrm{d}\mathbf{r}/\,\mathrm{d}t$, Week 9 gives

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{\gamma_u}, \qquad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}, \qquad u = |\mathbf{u}|.$$

Dividing a four-displacement by this invariant time gives the **four-velocity**. Multiplying by the constant rest mass m gives the **four-momentum**:

$$\boxed{U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma_u(c, \mathbf{u}), \qquad p^\mu = mU^\mu = (\mathcal{E}/c, \mathbf{p}).} \tag{3.1}$$

Here $\mathcal{E} = \gamma_u mc^2$ is total particle energy and $\mathbf{p} = \gamma_u m\mathbf{u}$ is ordinary three-momentum. Every component of U^μ has velocity units; every component of p^μ has momentum units.



divide displacement by its own $\Delta\tau$
then multiply by rest mass m

The clock supplies the same proper-time interval in every frame. The coordinate displacement has different components, so its ratio to proper time is a four-vector.

One invariant gives the energy–momentum relation

Since $U_\mu U^\mu = \gamma_u^2(c^2 - u^2) = c^2$, we have $p_\mu p^\mu = m^2c^2$. Expanding the components gives

$$\boxed{\mathcal{E}^2 = p^2c^2 + m^2c^4, \qquad K = \mathcal{E} - mc^2.} \tag{3.2}$$

K is kinetic energy. The nonzero rest energy mc^2 is included in $\mathcal{E}$.

Example — a particle initially at rest in the laboratory

What energy and momentum does S' assign to a particle at rest in S?

Solution. Start from $p^\mu = (mc, 0, 0, 0)$ and apply the boost:

$$\boxed{\mathcal{E}' = \gamma mc^2, \qquad p'_x = -\gamma mv, \qquad K' = (\gamma - 1)mc^2.}$$

The particle moves left in S'. Check that $\mathcal{E}'^2 - c^2p'^2_x = m^2c^4$; its rest mass has not changed.

Check / physical lesson

γ_u describes the particle's motion in a chosen frame; γ describes the relative motion of the two frames. They are equal only in special examples. The invariant $U_\mu U^\mu = c^2$ does not mean the particle travels at speed c.

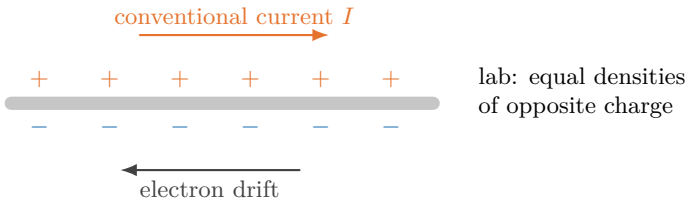
Next step. Apply the same transformation to charge density and current density.

4 Four-current: when a neutral wire looks charged

We now apply the same boost to the sources. A group of identical charges moving together has proper charge density ρ_0, measured in its rest frame. To measure the moving group's length, use simultaneous laboratory endpoints: $\Delta t = 0$ in the Lorentz transformation gives $\ell_0 = \gamma_u \ell$. The transverse area is unchanged, so the same charge occupies volume V_0/γ_u. Hence $\rho = \gamma_u \rho_0$ and $\mathbf{J} = \rho \mathbf{u}$, which combine as $J^\mu = \rho_0 U^\mu$. Add the contributions from different moving species to get the general four-current:

$$\boxed{J^\mu = (c\rho, \mathbf{J}), \qquad \rho' = \gamma\left(\rho - \frac{vJ_x}{c^2}\right), \quad J'_x = \gamma(J_x - v\rho).} \tag{4.1}$$

The transverse components are unchanged. Every entry of J^μ has units of current per unit area. A material with several species need not have one common velocity.



Charges of opposite sign can cancel in density while their currents do not cancel. The drawing shows a stationary positive lattice and drifting electrons.

Example — boost along a neutral current-carrying wire

A long uniform wire along x has lab line density $\lambda = 0$ and conventional current $I > 0$. What line density and current does the moving observer measure?

Solution. Integrate the density and current transformations across the wire. Its transverse area does not change in the boost, so $(c\lambda, I)$ transforms like $(c\rho, J_x)$. Thus

$$\boxed{\lambda' = -\gamma \frac{vI}{c^2}, \qquad I' = \gamma I.}$$

For $v > 0$, this observer sees negative net charge per unit length. In contrast, boosting a stationary charged line ($I = 0$) gives $\lambda' = \gamma\lambda$ and $I' = -\gamma v \lambda$.

Physical meaning

A fixed section of a current-carrying wire is open to charges moving through its ends. Its density at one time can change with frame. The total charge of an isolated finite system is invariant.

Next step. Write charge conservation using the same four-component notation.

5 Spacetime derivatives: read a conservation law

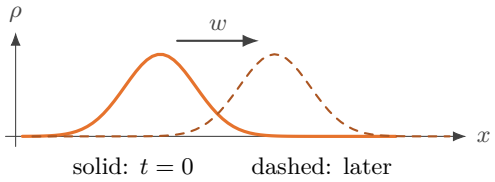
The field equations compare changes in time and space. To see which derivative carries which sign, start with a scalar h and its change between neighbouring events:

$$\mathrm{d}h = \partial_t h \, \mathrm{d}t + \nabla h \cdot \mathrm{d}\mathbf{r} = (\partial_\mu h) \, \mathrm{d}x^\mu .$$

This fixes the **four-gradient** with a lower index. Raising the index changes the signs of its three spatial entries:

$$\boxed{\partial_\mu = (c^{-1}\partial_t, \nabla), \qquad \partial^\mu = (c^{-1}\partial_t, -\nabla).} \tag{5.1}$$

A contraction of two derivatives yields the invariant wave operator $\Box = \partial_\mu \partial^\mu = c^{-2}\partial_t^2 - \nabla^2$. It is the negative of the operator $\nabla^2 - c^{-2}\partial_t^2$ used in Week 8.



A charge profile carried to the right. A point on its rising spatial slope sees its local density decrease: the material passing next comes from farther left.

Example — continuity for a moving packet of charge

Let a distribution, uniform across each transverse plane, move at constant speed w with $\rho = f(x - wt)$ and $J_x = w f(x - wt)$. Check local charge conservation.

Solution. With $J^\mu = (c\rho, J_x, 0, 0)$, *contract* the lower derivative with the upper current:

$$\partial_\mu J^\mu = \frac{1}{c}\partial_t(c\rho) + \partial_x J_x = -wf' + wf' = 0.$$

Here f' is the derivative of the profile with respect to $x - wt$. The ordinary continuity equation is therefore $\boxed{\partial_\mu J^\mu = 0}$. This is a scalar statement: if the divergence vanishes in the lab it vanishes in every inertial frame.

Check / physical lesson

The plus sign on $\nabla \cdot \mathbf{J}$ comes from using $\partial_\mu J^\mu$. Raising the derivative and leaving the current index up as well would not give the required scalar contraction.

Next step. Group the scalar and vector potentials, whose derivatives give the fields.

6 Four-potential: get a magnetic field from a scalar potential

A source that is electrostatic in its own frame may be moving in another. To predict the fields there, group the scalar and vector potentials into $A^\mu = (\Phi/c, \mathbf{A})$. The scalar potential Φ and vector potential $\mathbf{A}$ determine $\mathbf{E} = -\nabla\Phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Dividing Φ by c gives it the same units as $\mathbf{A}$. For the standard boost along x,

$$\boxed{\Phi' = \gamma(\Phi - vA_x), \qquad A'_x = \gamma(A_x - v\Phi/c^2), \qquad A'_y = A_y, \quad A'_z = A_z.}$$

The grouping can be tested: its derivatives must predict the measured fields, and in Lorenz gauge its source is the four-current (Sec. 16). This is a boost of a chosen potential. The primed functions must be evaluated at the same physical event, expressed in primed coordinates.



The gap field points up. The primed observer moves to the right in the lab. A boost mixes the potential components; their derivatives predict the new magnetic component.

Example — a capacitor moving parallel to its plates

In the uniform interior, take $\Phi = -E_0 y$ and $\mathbf{A} = 0$, with $E_0 > 0$. The plates are at rest in S and the boost is along $+x$. Recover both primed fields directly from the potentials.

Solution. Since $y' = y$ and the potentials have no x or t dependence, $\Phi' = -\gamma E_0 y'$ and $A'_x = +\gamma v E_0 y'/c^2$. Consequently

$$E'_y = -\partial_{y'}\Phi' = \gamma E_0, \qquad B'_z = -\partial_{y'}A'_x = -\gamma v E_0/c^2.$$

The minus sign in B'_z comes from the curl, not from the direction of the electric field.

Physical meaning

A **boost** changes the observer. A **gauge change** relabels the potentials for the *same* observer and leaves its measured fields unchanged. We use a bar, not a prime, for a gauge change below.

Next step. Learn to read a single matrix containing the six measured field components.

7 The field tensor: read the matrix before using it

We now arrange the six measured field components into one object, the **electromagnetic field tensor** $F^{\mu\nu}$. All entries refer to the *same event*. The first index labels the row; the second labels the column. The labels $0, 1, 2, 3$ mean ct, x, y, z. In particular, 0 does not mean $t = 0$, and the superscripts are not powers.

Key relations

With our metric $(+, -, -, -)$, the matrix is

$\mu\backslash\nu$	0 (ct)	1 (x)	2 (y)	3 (z)
0 (ct)	0	$-E_x/c$	$-E_y/c$	$-E_z/c$
1 (x)	E_x/c	0	$-B_z$	B_y
2 (y)	E_y/c	B_z	0	$-B_x$
3 (z)	E_z/c	$-B_y$	B_x	0

Violet entries contain electric fields; blue entries contain magnetic fields. The factor $1/c$ gives every entry the same SI units as B, namely tesla.

A time–space pair stores an electric component: $F^{02} = -E_y/c$. A pair of spatial directions stores the magnetic field normal to their plane: $F^{12} = -B_z$ belongs to the xy plane. The signs will follow directly from the potentials on the next page.

Physical meaning

Six field values, not sixteen new quantities. The matrix is **antisymmetric**, $F^{\nu\mu} = -F^{\mu\nu}$. Its four diagonal entries vanish and its six lower-triangle entries repeat the upper triangle with opposite signs. A zero diagonal does not mean zero force. The tensor stores the fields; charge and velocity determine the force.

Example — recover the fields from two matrix entries

Suppose the only nonzero entries above the diagonal are $F^{03} = -a$ and $F^{12} = -b$, where $a, b > 0$ have units of tesla. What fields are present?

Solution. The matrix gives $-E_z/c = -a$ and $-B_z = -b$, so

$$\boxed{\mathbf{E} = ca\hat{\mathbf{z}}, \qquad \mathbf{B} = b\hat{\mathbf{z}}.}$$

Both fields point along $+z$. The reversed entries are $F^{30} = a$ and $F^{21} = b$; they are not additional fields.

Method / strategy

Check the index positions before copying a sign. Lowering both indices means $F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}$. It reverses time–space entries but leaves spatial–spatial entries unchanged:

$$F_{02} = -F^{02} = +E_y/c, \qquad F_{12} = F^{12} = -B_z.$$

This changes the component notation, not the observer or the measured fields.

Next step. Construct the entries from the potentials and see why the signs fit together.

8 Build the tensor: one derivative for each field entry

We now derive the matrix from the familiar relations $\mathbf{E} = -\nabla\Phi - \partial_t\mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. With $A^\mu = (\Phi/c, \mathbf{A})$ and $\partial^\mu = (c^{-1}\partial_t, -\nabla)$, combine two derivatives as

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.} \tag{8.1}$$

Interchanging μ and ν reverses the sign automatically.

One electric entry and one magnetic entry

For the time–y pair, $\partial^0 = c^{-1}\partial_t$, $\partial^2 = -\partial_y$, $A^0 = \Phi/c$, and $A^2 = A_y$. Thus

$$F^{02} = \frac{1}{c}\partial_t A_y - (-\partial_y)\frac{\Phi}{c} = \frac{\partial_t A_y + \partial_y \Phi}{c} = -\frac{E_y}{c}.$$

For the x–y pair, both derivatives carry a spatial minus sign:

$$F^{12} = -\partial_x A_y + \partial_y A_x = -(\nabla \times \mathbf{A})_z = -B_z.$$

The other entries follow by the same calculation with x, y, z exchanged.

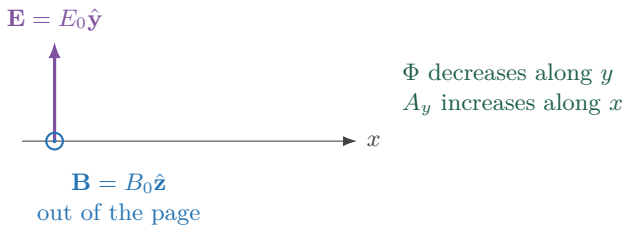
Example — construct the whole matrix for crossed fields

Take static potentials $\Phi = -E_0 y$ and $\mathbf{A} = B_0 x\hat{\mathbf{y}}$, where $E_0, B_0 > 0$. Find the fields and the nonzero tensor entries.

Solution. Their derivatives give $\mathbf{E} = E_0\hat{\mathbf{y}}$ and $\mathbf{B} = B_0\hat{\mathbf{z}}$. In the order $(0, 1, 2, 3)$,

$$[F^{\mu\nu}] = \begin{pmatrix} 0 & 0 & -E_0/c & 0 \\ 0 & 0 & -B_0 & 0 \\ E_0/c & B_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Two independent field components produce two antisymmetric pairs.



Different spatial derivatives of the potentials produce the two measured fields.

Check / physical lesson

A gauge change leaves every entry fixed. For a smooth scalar function $\chi(x, t)$, set $\bar{A}^\mu = A^\mu - \partial^\mu\chi$. The extra contribution is $-\partial^\mu\partial^\nu\chi + \partial^\nu\partial^\mu\chi = 0$ because mixed derivatives commute for smooth χ. The bar denotes a gauge change for one observer; a prime denotes a different observer.

Next step. Transform both tensor indices to find the fields measured by a moving observer.

9 Boost the fields: why electric and magnetic parts mix

We now transform the matrix from Eq. (8.1). Each index is transformed once:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\delta}F^{\alpha\delta}, \qquad F' = \Lambda F \Lambda^{\mathsf{T}}.$$

The perpendicular directions y and z do not change under our x boost, but the 0 and 1 labels mix. This is what mixes electric and magnetic entries.

Follow one pair of entries

For $\nu = 2$ the only terms in F'^{02} are

$$-E'_y/c = F'^{02} = \gamma(F^{02} - \beta F^{12}) = \gamma(-E_y/c + \beta B_z).$$

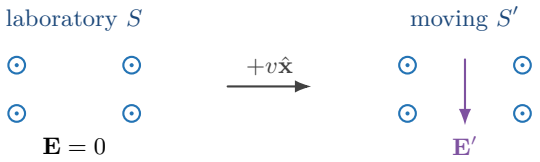
Doing the same for $F'^{12} = -B'_z$ gives the coupled pair

$$\boxed{E'_y = \gamma(E_y - vB_z), \qquad B'_z = \gamma(B_z - vE_y/c^2).}$$

The other pair is $E'_z = \gamma(E_z + vB_y)$ and $B'_y = \gamma(B_y + vE_z/c^2)$. The parallel components E_x, B_x do not change.

For any boost velocity $\mathbf{v}$, split each field into parts parallel and perpendicular to $\mathbf{v}$. The parallel parts stay fixed; the perpendicular parts obey

$$\boxed{\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}), \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E}/c^2).} \tag{9.1}$$



Both observers measure an outward magnetic field. Only the moving observer measures a downward electric field. Dots indicate direction; their size does not encode field strength.

Example — a magnet seen by a moving charge

In the lab let $\mathbf{E} = 0$, $\mathbf{B} = B_0\hat{\mathbf{z}}$. A charge momentarily moves with the frame S', whose velocity is $+v\hat{\mathbf{x}}$. What fields act in its instantaneous rest frame?

Solution. Since $\hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}$, Eq. (9.1) gives

$$\boxed{\mathbf{E}' = -\gamma v B_0 \hat{\mathbf{y}}, \qquad \mathbf{B}' = \gamma B_0 \hat{\mathbf{z}}.}$$

The charge is at rest in S', so its force there is electric. In the lab the same charge feels a magnetic force along the same transverse direction for $q > 0$. Section 13 compares their magnitudes.

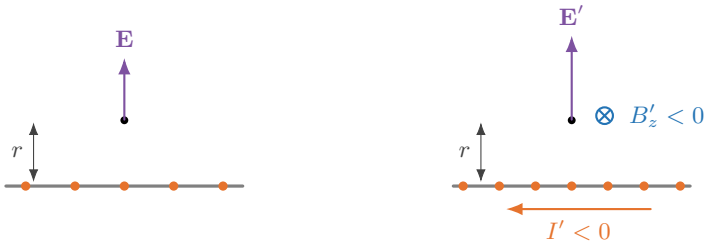
Method / strategy

Write the velocity of the *primed frame relative to the unprimed frame* first. To transform back, interchange primes and reverse $\mathbf{v}$. An observer change compares fields at the same event, not at two unrelated points.

Next step. Check these rules independently by transforming a source and solving for its fields.

10 The charged line: two ways to predict its fields

Return to the positive line charge λ at rest along the x axis. At a point a distance r on the $+y$ side, cylindrical symmetry and Gauss's law give $E_y = \lambda/(2\pi\varepsilon_0 r)$ and $\mathbf{B} = 0$. We now obtain the primed fields by two independent routes.



rest line moving line

On the upper side of the line, current to the left produces magnetic field into the page. The transverse distance r is the same in the two frames.

Example — one result from a boost and from an Ampèrian loop

Find E'_y and B'_z for an observer moving along $+x$.

Solution. **Transform the fields.** The boost of the lab solution gives

$$E'_y = \gamma \frac{\lambda}{2\pi\varepsilon_0 r}, \qquad B'_z = -\frac{\gamma v}{c^2}\frac{\lambda}{2\pi\varepsilon_0 r}.$$

Transform the source. Equation (4.1) gives $\lambda' = \gamma\lambda$, $I' = -\gamma v\lambda$ for a line at rest in S. The infinite uniform line is steady in S' too. Applying Gauss's law to a cylinder and Ampère's law to a circle gives

$$E'_y = \frac{\lambda'}{2\pi\varepsilon_0 r}, \qquad B'_z = \frac{\mu_0 I'}{2\pi r}.$$

These agree including signs, since $1/(\varepsilon_0 c^2) = \mu_0$.

Check / physical lesson

A positive test charge momentarily at rest relative to the line moves at $-v\hat{\mathbf{x}}$ in S'. There its magnetic force points toward the line and partly cancels the outward electric force:

$$f'_y = q(E'_y + vB'_z) = q\gamma E_y(1-\beta^2) = qE_y/\gamma.$$

The three-force is frame dependent. The two observers agree on the trajectory once both position *and time* are transformed.

Next step. Look for field combinations on which both observers agree.

11 Field invariants: what cannot change with the observer?

Two quantities can rule out an impossible field transformation before we calculate a boost. At the same event, define

$$\boxed{I_1 = E^2 - c^2B^2, \qquad I_2 = \mathbf{E}\cdot\mathbf{B}.} \tag{11.1}$$

Both have the same values under Lorentz boosts. The first compares the electric and magnetic strengths in the same units. The second tests whether the fields are perpendicular.

See the invariant in one coupled pair

For the (E_y, cB_z) pair, put $a = E_y$ and $b = cB_z$. Their primed values are $a' = \gamma(a - \beta b)$ and $b' = \gamma(b - \beta a)$. Therefore

$$a'^2 - b'^2 = \gamma^2(1-\beta^2)(a^2 - b^2) = a^2 - b^2.$$

The (E_z, cB_y) pair gives the same cancellation with the opposite sign of β; the parallel components stay fixed. This proves invariance of I_1. Section 19 gives a compact proof for I_2 as well.

Measured at one event	What a different inertial observer can achieve
$I_2 \neq 0$	Neither field can vanish: their scalar product would then be zero.
$I_2 = 0,\ I_1 > 0$	A purely electric frame exists.
$I_2 = 0,\ I_1 < 0$	A purely magnetic frame exists.
$I_1 = I_2 = 0$, nonzero field	A **null field**: neither field can vanish in any inertial frame.

Example — can a moving observer stop a light wave?

At a nonzero crest of a vacuum plane wave, $\mathbf{E} \perp \mathbf{B}$ and $E = cB$. Could a boost make it a purely electric field?

Solution. Its invariants are both zero. If $B' = 0$, the first invariant would require $E' = 0$ too. But the boost has an inverse: zero primed fields would imply zero lab fields, contradicting the nonzero crest. The same argument excludes a purely magnetic frame.

Physical meaning

The zero field also gives two zero invariants, so always check that the field is nonzero. A plane wave is one example of a null field. A local measurement of these two numbers does not determine the wavefront shape or prove that the whole field is a plane wave.

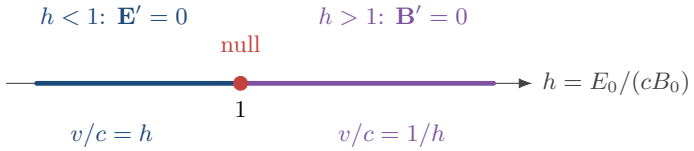
Next step. Use the invariants to choose a frame that makes a crossed-field problem simpler.

12 Choose the useful frame: solve the crossed-field problem

For $\mathbf{E} = E_0\hat{\mathbf{y}}$, $\mathbf{B} = B_0\hat{\mathbf{z}}$, $E_0, B_0 > 0$, the direction of $\mathbf{E} \times \mathbf{B}$ is $+x$. Boost in that direction. The two relevant equations are

$$E'_y = \gamma(E_0 - vB_0), \qquad B'_z = \gamma(B_0 - vE_0/c^2).$$

Each can be set to zero algebraically. Only a solution with $v < c$ describes an inertial observer.



Decision line for perpendicular nonzero fields. At $h = 1$ both prescriptions would require $v = c$.

Example — find the boost and the surviving field

Find the simplest field seen by any observer, for both orderings of E_0 and cB_0.

Solution. If $E_0 < cB_0$, choose $v = E_0/B_0$. Then $E'_y = 0$, and the invariant or direct substitution gives

$$\boxed{B'_z = \sqrt{B_0^2 - E_0^2/c^2}.}$$

A charge at rest in this frame feels no force; in the laboratory it moves at E_0/B_0 and the electric and magnetic forces cancel. If $E_0 > cB_0$, instead choose $v = c^2B_0/E_0$. Now $B'_z = 0$ and

$$\boxed{E'_y = \sqrt{E_0^2 - c^2B_0^2}.}$$

The positive square roots follow from the stated orientations. In either case the field left over is smaller than its lab counterpart.

Check / physical lesson

The criteria are local. For uniform crossed fields a single boost works everywhere; if the field ratio or direction varies in space, the simplifying boost may differ from point to point. It is not unique: a further boost along the surviving field leaves that field pure.

Next step. Return to the moving charge and calculate both force and power in the same notation.

13 Four-force: read the matrix as force and power

We now use the field matrix to calculate how a charge's energy and momentum change. For charge q with constant rest mass, the **four-force** is

$$\boxed{K^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}U_\nu, \qquad U_\nu = \gamma_u(c, -u_x, -u_y, -u_z).} \tag{13.1}$$

Hold the row μ fixed and sum over the column ν. Here K^μ is a four-vector; the ordinary three-force is $\mathbf{f} = \mathrm{d}\mathbf{p}/\,\mathrm{d}t$. Thus $K^i = \gamma_u f_i$.

Every term in the x row

Since $F^{11} = 0$, the nonzero contributions are

$$\begin{aligned} K^1 &= q\big(F^{10}U_0 + F^{12}U_2 + F^{13}U_3\big) \\ &= q\gamma_u\big[(E_x/c)c + (-B_z)(-u_y) + B_y(-u_z)\big] \\ &= q\gamma_u\big[E_x + u_yB_z - u_zB_y\big]. \end{aligned}$$

The first term is electric. The other two are the x component of $\mathbf{u}\times\mathbf{B}$. The minus signs in U_i are needed for this result. The time row similarly gives $K^0 = q\gamma_u\mathbf{E}\cdot\mathbf{u}/c$. Using $p^0 = \mathcal{E}/c$ and $\mathrm{d}/\,\mathrm{d}\tau = \gamma_u\,\mathrm{d}/\,\mathrm{d}t$, all four rows give

$$\boxed{\mathbf{f} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B}), \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = q\mathbf{E}\cdot\mathbf{u}.}$$

The time row gives power. A magnetic force does no work because it is perpendicular to the instantaneous velocity.

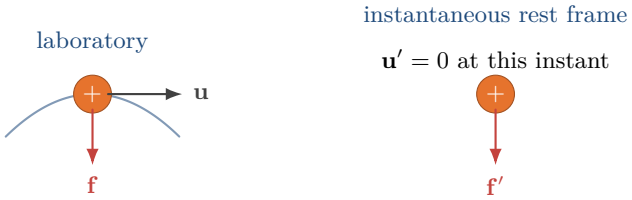
Example — finish Week 9's moving-charge question

In the laboratory, a positive charge has instantaneous velocity $v\hat{\mathbf{x}}$ in $\mathbf{B} = B_0\hat{\mathbf{z}}$, with $\mathbf{E} = 0$. Compare forces in S and its instantaneous rest frame S'.

Solution. The lab gives $f_y = -qvB_0$. The field boost gives $E'_y = -\gamma vB_0$; since $\mathbf{u}' = 0$ at this event,

$$\boxed{f'_y = qE'_y = -q\gamma vB_0 = \gamma f_y.}$$

Independently, an x boost leaves $p'_y = p_y$, while on the particle's path at this instant $\mathrm{d}t' = \mathrm{d}t/\gamma$. Hence $\mathrm{d}p'_y/\,\mathrm{d}t' = \gamma\,\mathrm{d}p_y/\,\mathrm{d}t$, in agreement.



Arrows show force directions, not relative magnitudes. A charge can have zero velocity at one instant and still have nonzero acceleration.

S' is one fixed inertial frame matching the charge's velocity at the chosen event. The charge accelerates away from rest in that frame immediately; it is not attached to S' throughout its curved motion.

Next step. Use the same matrix to recover Maxwell's equations and their sources.

14 Maxwell with sources: differentiate one column at a time

We now recover Gauss's law and the Ampère–Maxwell law from one equation. With $J^\nu = (c\rho, J_x, J_y, J_z)$ and vacuum permeability μ_0, write

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu.} \tag{14.1}$$

Here ν is a **free index**: choose one value and keep it fixed. The repeated μ is summed from 0 to 3:

$$\partial_0 F^{0\nu} + \partial_1 F^{1\nu} + \partial_2 F^{2\nu} + \partial_3 F^{3\nu} = \mu_0 J^\nu.$$

Thus we hold a *column* fixed and differentiate down the rows. The force law instead holds a row fixed and sums over columns. Both sides of Eq. (14.1) are four-vectors, so the equation keeps the same form after a boost. This is **covariance**.

The time column gives charge; a spatial column gives current

For $\nu = 0$, $F^{00} = 0$ and $F^{i0} = E_i/c$:

$$\frac{\partial_x E_x + \partial_y E_y + \partial_z E_z}{c} = \mu_0 c\rho \quad \Longrightarrow \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

For $\nu = 1$, the nonzero entries are $F^{01} = -E_x/c$, $F^{21} = B_z$, and $F^{31} = -B_y$. They give

$$\underbrace{\partial_y B_z - \partial_z B_y}_{(\nabla\times\mathbf{B})_x} - \frac{1}{c^2}\partial_t E_x = \mu_0 J_x.$$

Columns 2 and 3 give the other components of $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + c^{-2}\partial_t \mathbf{E}$.

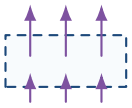
Example — which sources support this local field?

In a region, let $\mathbf{E} = (ay + bt)\hat{\mathbf{y}}$ and $\mathbf{B} = 0$. The constants $a = \partial_y E_y$ and $b = \partial_t E_y$ describe a spatial gradient and a time rate. Find the required charge and current.

Solution. Gauss's law gives $\rho = \varepsilon_0 a$. With no magnetic curl, the material current cancels the displacement term:

$$\boxed{\rho = \varepsilon_0 a, \qquad \mathbf{J} = -\varepsilon_0 b\,\hat{\mathbf{y}}.}$$

The field is curl-free, so Faraday's law also holds. This is a local model; it need not describe the field out to infinity.



net outward electric flux



equal and opposite contributions

For $a, b > 0$ and $E_y > 0$, the stronger upper field signals positive charge. The upward displacement term and downward material current give zero magnetic curl.

Physical meaning

Charge conservation is built in. Applying ∂_ν gives $\mu_0 \partial_\nu J^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0$: commuting derivatives are symmetric in μ, ν, while $F^{\mu\nu}$ is antisymmetric. The terms cancel in pairs.

Next step. Write the remaining two Maxwell laws using the same column-reading rule.

15 The other Maxwell laws: rearrange the same six fields

We now write magnetic Gauss's law and Faraday's law in the same compact form. These laws hold even where charges and currents are present. Build the **dual tensor** $\widetilde{F}$ from the field matrix in Sec. 7 by the substitutions

$$\mathbf{E}/c \longmapsto \mathbf{B}, \qquad \mathbf{B} \longmapsto -\mathbf{E}/c.$$

The tilde means a rearrangement of the same six field values, not a new field or a new observer. This rule is enough to use the dual; optional Sec. 19 derives its index formula.

$$\boxed{\partial_\mu \widetilde{F}^{\mu\nu} = 0.} \tag{15.1}$$

This is called the **homogeneous** Maxwell equation because its right-hand side is zero. Here "homogeneous" does *not* mean spatially uniform or charge-free.

Read the time column and the z column

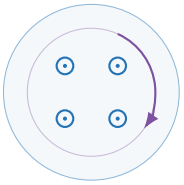
Column 0 contains $\widetilde{F}^{i0} = B_i$, so it gives $\nabla \cdot \mathbf{B} = 0$. In column 3, the entries are

$$\widetilde{F}^{03} = -B_z, \qquad \widetilde{F}^{13} = -E_y/c, \qquad \widetilde{F}^{23} = E_x/c.$$

Differentiating them with $\partial_\mu = (c^{-1}\partial_t, \nabla)$ gives

$$-\frac{1}{c}\left(\partial_t B_z + \partial_x E_y - \partial_y E_x\right) = 0.$$

This is the z component of $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$. Columns 1 and 2 give its other two components.



outward magnetic field increasing

clockwise electric circulation

The circle lies inside the region of uniform magnetic field. For increasing outward flux, Faraday's minus sign requires clockwise circulation.

Example — read induction from the dual equation

Inside a long ideal solenoid, let $B_z = B_0 + \alpha t$, where α is a constant rate of increase. Check the induced field $\mathbf{E} = (\alpha/2)(y\hat{\mathbf{x}} - x\hat{\mathbf{y}})$.

Solution. The three terms in the z equation are

$$\partial_t B_z = \alpha, \qquad \partial_x E_y = -\alpha/2, \qquad \partial_y E_x = \alpha/2.$$

They give $\alpha - \alpha/2 - \alpha/2 = 0$. Around a circle of radius r inside the solenoid, $E_\phi = -\alpha r/2$, so

$$\oint \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\pi r^2 \alpha = -\frac{\mathrm{d}}{\mathrm{d}t}(\pi r^2 B_z).$$

The component equation and the flux law give the same sign and magnitude.

Next step. Use the potential definition to turn Maxwell's equations into wave equations.

16 Lorenz gauge: from sources to travelling fields

We now close the chain from sources to potentials and then to measured fields. Insert Eq. (8.1) into Eq. (14.1) and commute derivatives:

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = \mu_0 J^\nu, \qquad \Box = \frac{1}{c^2}\partial_t^2 - \nabla^2.$$

The box is a derivative operator, not a new field. It has the opposite sign to $\nabla^2 - c^{-2}\partial_t^2$ used in Week 8.

Key relations

Choose the **Lorenz gauge**,

$$\partial_\mu A^\mu = \frac{1}{c^2}\partial_t\Phi + \nabla\cdot\mathbf{A} = 0.$$

The remaining equations are

$$\boxed{\Box A^\nu = \mu_0 J^\nu, \qquad \Box\Phi = \frac{\rho}{\varepsilon_0}, \qquad \Box\mathbf{A} = \mu_0\mathbf{J}.} \tag{16.1}$$

The time component uses $A^0 = \Phi/c$, $J^0 = c\rho$, and $\mu_0 c^2 = 1/\varepsilon_0$; the spatial components give the vector equation.

The gauge condition is a scalar, so a boost preserves it. Choosing a gauge alone does not select a unique solution: initial and boundary data still matter. A **retarded** solution makes a source change arrive at distance R after the travel time R/c.

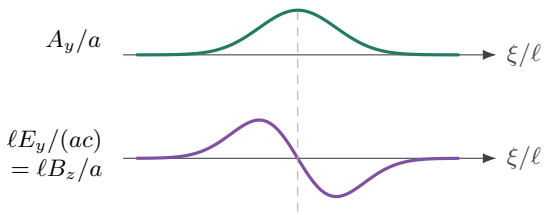
Example — build the Week 9 wave from one potential

Take $\Phi = 0$ and $\mathbf{A} = af(x-ct)\hat{\mathbf{y}}$, where a is a constant potential amplitude and f is a smooth dimensionless profile. Find the fields and check that no local source is required.

Solution. The divergence of $\mathbf{A}$ is zero: its y component depends on x, t, not on y. Also $\Box A_y = 0$ by the chain rule, so Lorenz gauge and the source-free potential equation both hold. One derivative gives

$$\boxed{E_y = -\partial_t A_y = acf'(x-ct), \qquad B_z = \partial_x A_y = af'(x-ct).}$$

The prime on f means a derivative of its argument, not another frame. Thus $E_y = cB_z$, and the energy flows along $+x$.



For $\xi = x - ct$ and $f(\xi) = \exp[-(\xi/\ell)^2]$, the fields follow the potential's slope. They vanish at its maximum and change sign across it.

Check / physical lesson

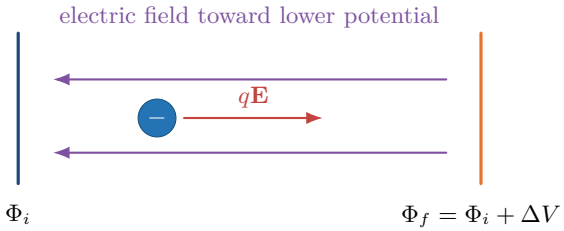
Lorenz names the gauge; **Lorentz** names the boost. A gauge change $\bar{A}^\mu = A^\mu - \partial^\mu\chi$ leaves the fields fixed. It preserves this gauge when $\Box\chi = 0$.

Bridge to Week 11. We can now describe one electromagnetic experiment in different inertial frames. Next we return to waves: their polarization and the boundary conditions at an interface determine what is reflected and transmitted.

OPTIONAL — Relativistic motion through an accelerating voltage
Further detail, following the main development.

17 Electric work: find momentum without solving the trajectory

We now use the energy–momentum relation to calculate how a voltage accelerates an electron. The result should work both for slow motion and when the speed approaches c.



The electron has charge $q = -e$, where $e > 0$. Its force opposes the electric field, so it gains speed toward higher potential.

Example — one symbolic solution for every accelerating voltage

An electron of rest mass m starts from rest and crosses a static potential difference $\Delta V = \Phi_f - \Phi_i > 0$. Neglect radiation and other energy losses. Find its final momentum magnitude p and speed u.

Solution. The electric work is $-q\Delta V = e\Delta V$. Thus the final kinetic energy is $K = e\Delta V$ and the total energy is $\mathcal{E} = mc^2 + e\Delta V$. Introduce the dimensionless ratio

$$\xi = \frac{e\Delta V}{mc^2}, \qquad \mathcal{E} = mc^2(1+\xi).$$

Using Eq. (3.2),

$$p^2c^2 = \mathcal{E}^2 - m^2c^4 = m^2c^4\big[(1+\xi)^2 - 1\big].$$

Since $\mathcal{E} = \gamma_u mc^2$ and $u = pc^2/\mathcal{E}$, the answer is

$$\boxed{\gamma_u = 1+\xi, \qquad p = mc\sqrt{\xi(2+\xi)}, \qquad \frac{u}{c} = \frac{\sqrt{\xi(2+\xi)}}{1+\xi}.} \tag{17.1}$$

Here K is scalar kinetic energy; the indexed K^μ in Sec. 13 denotes four-force.

Check / physical lesson

For $\xi \ll 1$, the formulas reduce to

$$p \simeq \sqrt{2me\Delta V}, \qquad u \simeq \sqrt{\frac{2e\Delta V}{m}}.$$

For every finite voltage, $1 - u^2/c^2 = (1+\xi)^{-2} > 0$. Increasing the voltage can keep increasing the energy and momentum, while the speed remains below c.

Physical meaning

The useful comparison is $e\Delta V$ against mc^2. It tells us when Newtonian kinetic energy is a good approximation. The work calculation itself remains $\Delta K = -q\Delta V$ for a static electric potential.

OPTIONAL — A moving observer measures a whole light pulse
Further detail, following the main development.

18 Doppler shift: connect field strength, duration, and energy

We now revisit Week 9's right-moving plane pulse. Its fields satisfy $B_z = E_y/c$, and its energy flux is $S_x = \varepsilon_0 c E_y^2$. What does an observer moving along $+x$ at speed $0 < v < c$ measure?

The same profile has a different argument

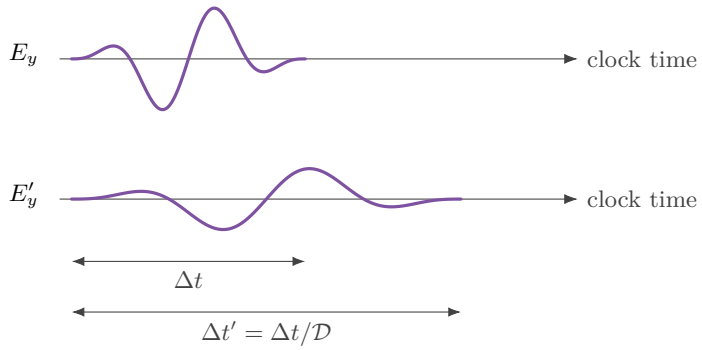
Define the positive Doppler factor

$$\mathcal{D} = \gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}, \qquad \beta = v/c.$$

Equation (9.1) gives $E'_y = \mathcal{D}E_y$ and $B'_z = \mathcal{D}B_z$ at the same event. The inverse coordinate transformation gives $x - ct = \mathcal{D}(x' - ct')$. Thus a profile $E_y = F(x - ct)$ becomes

$$\boxed{E'_y(x', t') = \mathcal{D}F\big(\mathcal{D}(x' - ct')\big).}$$

Its amplitude is smaller and its duration at a fixed detector is longer. For a sinusoidal carrier, the phase gives $\omega' = \mathcal{D}\omega$.



Two local detector traces with their start times aligned. The axes use common field and time scales. A receding observer sees fewer cycles per second, but the same total number of cycles in the pulse.

Example — why the pulse energy has only one Doppler factor

Compare the energy U and U' intercepted through the same transverse area A by stationary and receding ideal absorbers in separate experiments. Each absorber receives the entire pulse; recoil is negligible.

Solution. At matching positions within the two profiles, $S'_x = \mathcal{D}^2 S_x$ and $\mathrm{d}t' = \mathrm{d}t/\mathcal{D}$. The transverse area is unchanged, so

$$\boxed{U' = A\int S'_x\,\mathrm{d}t' = \mathcal{D}A\int S_x\,\mathrm{d}t = \mathcal{D}U.}$$

For a sinusoidal wave, intensity means cycle-averaged flux, $I = \langle S_x\rangle$, and $I' = \mathcal{D}^2 I$. Energy acquires one factor of $\mathcal{D}$ because the weaker flux acts for a longer time.

OPTIONAL — Reconstructing the dual tensor and its signs
Further detail, following the main development.

19 Complementary pairs: reconstruct the dual signs

We now derive the dual used in Sec. 15. It contains the same six fields, arranged by the two directions missing from each index pair. Fix the orientation by $\varepsilon^{0123} = +1$. The alternating symbol is zero if an index repeats and changes sign whenever two indices are exchanged.

$$\boxed{\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}.} \tag{19.1}$$

Here ρ and σ are summed index labels, each running from 0 to 3. They do not denote charge density in this formula. With the metric $(+,-,-,-)$, lowering both indices reverses the time–space entries but leaves the space–space entries unchanged.

selected pair	complementary pair	
$\mu\nu = 01$	$\rho\sigma = 23$	$\widetilde{F}^{01} = F_{23} = -B_x$
$\mu\nu = 02$	$\rho\sigma = 31$	$\widetilde{F}^{02} = F_{31} = -B_y$
$\mu\nu = 03$	$\rho\sigma = 12$	$\widetilde{F}^{03} = F_{12} = -B_z$

The cyclic order shown fixes the signs. Reversing either pair changes its sign. For the reverse pairing, $\widetilde{F}^{12} = F_{03} = E_z/c$.

Why the factor one-half is present

For $\widetilde{F}^{01}$, only the pairs 23 and 32 survive:

$$\widetilde{F}^{01} = \tfrac{1}{2}\big(\varepsilon^{0123}F_{23} + \varepsilon^{0132}F_{32}\big) = \tfrac{1}{2}\big(F_{23} + F_{23}\big) = F_{23}.$$

Each independent antisymmetric pair appears twice in the full sum.

Example — parallel fields cannot lose either part under a boost

Let $\mathbf{E} = E_0\hat{\mathbf{z}}$ and $\mathbf{B} = B_0\hat{\mathbf{z}}$, with both amplitudes nonzero. Evaluate $F_{\mu\nu}\widetilde{F}^{\mu\nu}$, summing both repeated indices.

Solution. The independent nonzero pairs are 03 and 12:

$$2\big[(E_0/c)(-B_0) + (-B_0)(E_0/c)\big] = -4E_0B_0/c.$$

For arbitrary fields, including the other two directions gives

$$\boxed{F_{\mu\nu}\widetilde{F}^{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}/c, \qquad F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2/c^2).}$$

All indices are contracted, so these quantities are unchanged by the Lorentz boosts used here. In the parallel example $\mathbf{E}\cdot\mathbf{B} \neq 0$; setting either primed field to zero would contradict this invariant.

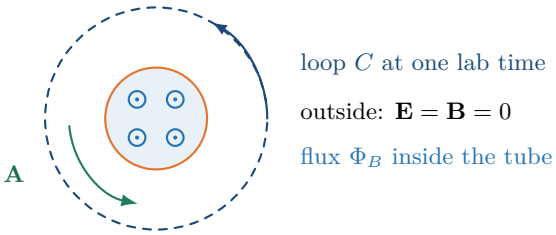
OPTIONAL — What a potential can record outside a solenoid
Further detail, following the main development.

20 A loop around flux: local fields and a global integral

We now ask whether zero field in a region always permits one gauge choice with zero potential throughout that region. Consider a static ideal infinite solenoid of radius R, with axial magnetic flux Φ_B. Outside it, $\mathbf{E} = \mathbf{B} = 0$ and therefore $F^{\mu\nu} = 0$. At radial distance $\varrho > R$, a useful potential is

$$\Phi = 0, \qquad \mathbf{A} = \frac{\Phi_B}{2\pi\varrho}\hat{\boldsymbol{\phi}},$$

where $\hat{\boldsymbol{\phi}}$ is the unit vector around the axis. Its exterior curl is zero because ϱA_ϕ is constant. Yet a counterclockwise circle C around positive axial flux obeys $\oint_C \mathbf{A}\cdot \mathrm{d}\boldsymbol{\ell} = \Phi_B$.



The loop cannot shrink to a point while staying outside the solenoid. A surface spanning it crosses the interior field; this is consistent with Stokes's theorem.

Example — keep the same closed path of events

Define $\mathcal{C} = \oint A_\mu \, \mathrm{d}x^\mu$ around the loop in the drawing. Find its value and check what a gauge change or an observer change does to it.

Solution. Lowering the potential index gives $A_\mu = (\Phi/c, -\mathbf{A})$. Thus

$$A_\mu \, \mathrm{d}x^\mu = \Phi \, \mathrm{d}t - \mathbf{A}\cdot \mathrm{d}\mathbf{r}.$$

The drawn loop has $\mathrm{d}t = 0$, so $\boxed{\mathcal{C} = -\Phi_B}$. A smooth, single-valued gauge function χ changes the integral by

$$\Delta\mathcal{C} = -\oint \partial_\mu\chi \, \mathrm{d}x^\mu = -\oint \mathrm{d}\chi = 0.$$

Under a boost, $A_\mu \, \mathrm{d}x^\mu$ is a scalar. The *same closed path of events* therefore has the same integral, although those events generally occur at different primed times.

Physical meaning

If a smooth, single-valued gauge made $\mathbf{A} = 0$ everywhere outside the tube, the loop integral would be zero, contradicting $\Phi_B \neq 0$. The potential can be removed on a small exterior patch, but not around the whole hole in one such gauge. This concerns the potential's global structure; there is no local classical electromagnetic force outside the static ideal tube.