Electrodynamics & Relativity

Week 8: Potentials and Field Conservation

Gauge freedom, energy, momentum, and stress

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Main idea. Week 7 completed Maxwell's equations. We now use potentials to organize their solutions, then calculate how fields store energy, carry momentum, and exert forces. The solenoid, capacitor, and coaxial geometry from the earlier weeks will connect these ideas through worked examples.

1 Potentials: keep the fields, simplify the description

The two Maxwell relations without explicit source terms, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$, are satisfied automatically by

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\Phi - \partial_t \mathbf{A}.} \tag{1.1}$$

Here $\Phi(\mathbf{r}, t)$ is the scalar potential and $\mathbf{A}(\mathbf{r}, t)$ is the vector potential. We use ∂_t for a time derivative at fixed position. The remaining Maxwell equations determine the potentials from the sources and boundary data.

1.1 Gauge freedom: change the description together

For a smooth scalar function $\chi(\mathbf{r}, t)$, the transformation

$$\boxed{\mathbf{A}' = \mathbf{A} + \nabla\chi, \qquad \Phi' = \Phi - \partial_t \chi} \tag{1.2}$$

leaves both fields unchanged. This is **gauge freedom**.

Check the cancellation

The magnetic field does not change because $\nabla \times \nabla\chi = 0$. The two changes in the electric field cancel:

$$\mathbf{E}' - \mathbf{E} = \nabla(\partial_t \chi) - \partial_t(\nabla\chi) = 0.$$

Changing $\mathbf{A}$ alone while forgetting the corresponding change of Φ would, in general, change the electric field.

Method / strategy

A **gauge condition** selects a convenient representative of the same physical fields. First choose a gauge, then solve for the potentials, and finally differentiate to obtain $\mathbf{E}$ and $\mathbf{B}$. A proposed solution must satisfy the source equations as well as the two relations already built into Eq. (1.1).

Physical meaning

All energy, momentum, and force calculations below use $\mathbf{E}$ and $\mathbf{B}$, so their results are independent of gauge. We use SI units and the vacuum constants ε_0, μ_0, with $c^{-2} = \mu_0\varepsilon_0$. Material polarization and magnetization are outside this week's scope.

2 A changing uniform field: let symmetry choose the potential

We first recover the induction result of Week 7 from a simple potential. In a region with axial field $\mathbf{B} = B(t)\hat{\mathbf{z}}$, use cylindrical radius ϱ and azimuthal unit vector $\hat{\boldsymbol{\phi}}$. Choose a circulating potential and no scalar-potential gradient.

Example — determine the coefficient, then obtain the electric field

Try $\mathbf{A} = a(t)\varrho\hat{\boldsymbol{\phi}}$ and $\Phi = 0$. Find $a(t)$ and the induced field.

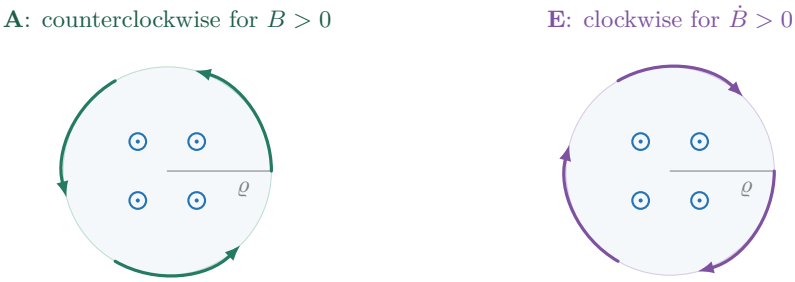
Solution. For this azimuthal potential,

$$(\nabla \times \mathbf{A})_z = \frac{1}{\varrho}\frac{\partial}{\partial \varrho}(\varrho A_\phi) = \frac{1}{\varrho}\frac{\partial}{\partial \varrho}(a\varrho^2) = 2a.$$

Matching the prescribed field gives $a = B/2$. Therefore

$$\boxed{\mathbf{A} = \frac{B(t)\varrho}{2}\hat{\boldsymbol{\phi}}, \qquad \mathbf{E} = -\frac{\dot{B}(t)\varrho}{2}\hat{\boldsymbol{\phi}}.} \tag{2.1}$$

For a linear ramp $B(t) = B_0 + \alpha t$, the electric field is $\mathbf{E} = -(\alpha\varrho/2)\hat{\boldsymbol{\phi}}$: it depends on the ramp rate α, not on the initial field B_0.



The potential records the magnetic field; its time derivative produces electric circulation. Blue dots denote magnetic field out of the page.

Check / physical lesson

A circle of radius ϱ gives

$$\oint \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = 2\pi\varrho E_\phi = -\pi\varrho^2\dot{B} = -\frac{\mathrm{d}}{\mathrm{d}t}(\pi\varrho^2 B),$$

which independently checks Faraday's law and the sign.

Check the assumed field profile. The fields have $\rho = \varepsilon_0\nabla\cdot\mathbf{E} = 0$ and $\nabla\times\mathbf{B} = 0$. Ampère–Maxwell therefore requires $\mathbf{J} = -\varepsilon_0\partial_t\mathbf{E} = (\varepsilon_0\varrho\ddot{B}/2)\hat{\boldsymbol{\phi}}$. A source-free region admits this exactly uniform profile for a linear ramp, where $\ddot{B} = 0$. For a general drive in a long solenoid, it is a slowly varying local approximation; rapid changes require spatial field corrections.

3 Equations for the potentials: what remains to be solved?

We now insert the potentials into the two Maxwell equations with sources. This shows exactly which terms a gauge choice can simplify.

Gauss's law and Ampère–Maxwell

From $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$,

$$\nabla^2\Phi + \partial_t(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}. \tag{3.1}$$

Use $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ in $\nabla \times \mathbf{B} = \mu_0\mathbf{J} + c^{-2}\partial_t\mathbf{E}$:

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J} - c^{-2}\nabla\partial_t\Phi - c^{-2}\partial_t^2\mathbf{A}.$$

Rearranging gives

$$\nabla^2\mathbf{A} - c^{-2}\partial_t^2\mathbf{A} - \nabla\underbrace{\left(\nabla \cdot \mathbf{A} + c^{-2}\partial_t\Phi\right)}_{G} = -\mu_0\mathbf{J}. \tag{3.2}$$

The symbol G is only a shorthand for the combination in brackets, not a new field.

3.1 Coulomb gauge: retain a Poisson equation

Choose $\nabla \cdot \mathbf{A} = 0$. The equations become

$$\boxed{\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}, \qquad \nabla^2\mathbf{A} - c^{-2}\partial_t^2\mathbf{A} = -\mu_0\mathbf{J} + c^{-2}\nabla\partial_t\Phi.} \tag{3.3}$$

The scalar equation has the same spatial form as in Weeks 3 and 4. Its charge density and boundary data may still depend on time. The vector equation retains the scalar-potential term.

Example — impose Coulomb gauge explicitly

For constant B_0 and β, consider $\mathbf{A} = B_0 x\hat{\mathbf{y}} + \beta x\hat{\mathbf{x}}$ and $\Phi = 0$. Both terms are allowed, but $\nabla \cdot \mathbf{A} = \beta$. Find a gauge with zero divergence.

Solution. We need $\nabla^2\chi = -\nabla \cdot \mathbf{A} = -\beta$. A simple choice is $\chi = -\beta x^2/2$, giving

$$\mathbf{A}' = \mathbf{A} + \nabla\chi = B_0 x\hat{\mathbf{y}}, \qquad \Phi' = 0.$$

Now $\nabla \cdot \mathbf{A}' = 0$, while both choices give $\mathbf{B} = B_0\hat{\mathbf{z}}$ and $\mathbf{E} = 0$.

Physical meaning

An instantaneous Poisson equation for Φ does not imply instantaneous transmission of a physical signal. The measured field is the full combination $-\nabla\Phi - \partial_t\mathbf{A}$. Its two terms must be solved consistently with the same sources and initial and boundary conditions. A gauge choice changes how this field is represented, not how a disturbance propagates.

4 Lorenz gauge: give both potentials a wave equation

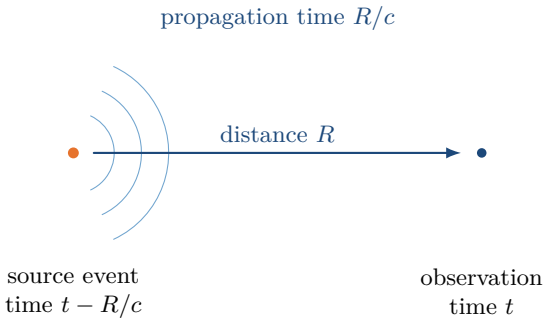
We now choose the gauge that removes the coupled gradient term in Eq. (3.2):

$$\boxed{G \equiv \nabla \cdot \mathbf{A} + c^{-2}\partial_t \Phi = 0.} \tag{4.1}$$

This is the **Lorenz gauge**. It gives two equations of the same form:

$$\boxed{\begin{aligned} \nabla^2 \Phi - c^{-2}\partial_t^2 \Phi &= -\rho/\varepsilon_0, \\ \nabla^2 \mathbf{A} - c^{-2}\partial_t^2 \mathbf{A} &= -\mu_0 \mathbf{J}. \end{aligned}} \tag{4.2}$$

The scalar equation follows by putting $\nabla \cdot \mathbf{A} = -c^{-2}\partial_t \Phi$ into Eq. (3.1). The vector equation follows immediately from $G = 0$. Each Cartesian component of $\mathbf{A}$ has a wave equation with the same speed c. The sources must obey charge conservation, $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$.



For the causal solution, a disturbance reaches a distant observation point after a delay. The wave equations still require initial and boundary conditions; the gauge condition alone does not select that solution.

4.1 A condition can leave some gauge freedom

Under Eq. (1.2),

$$G' = G + \nabla^2 \chi - c^{-2}\partial_t^2 \chi.$$

Thus a transformation preserves Lorenz gauge when

$$\boxed{\nabla^2 \chi - c^{-2}\partial_t^2 \chi = 0.} \tag{4.3}$$

This remaining choice is called **residual gauge freedom**.

Example — a whole family of allowed gauge functions

Let $\chi = \chi_0 f(z - ct)$, where f is any twice-differentiable function. Does it preserve Lorenz gauge?

Solution. Write $s = z - ct$. Then

$$\nabla^2 \chi = \chi_0 f''(s), \qquad c^{-2}\partial_t^2 \chi = \chi_0 f''(s).$$

Their difference is zero. The transformed potentials satisfy the same gauge condition and produce exactly the same fields. A wave-shaped gauge function is not an additional physical wave.

5 A wave from one potential: recover both fields

We now construct a physical wave, using the plane-wave geometry introduced in Week 7. Take a source-free region and choose

$$\Phi = 0, \qquad \mathbf{A} = A_0 \sin(kz - \omega t)\hat{\mathbf{x}}, \qquad k > 0,\ \omega > 0.$$

Here A_0 is a constant amplitude and k is the wave number. Since A_x depends on z, not x, its divergence is zero. These potentials satisfy both Coulomb and Lorenz gauge.

Example — derive the speed, directions, and relative amplitudes

Determine the condition on ω and k, then calculate $\mathbf{E}$ and $\mathbf{B}$.

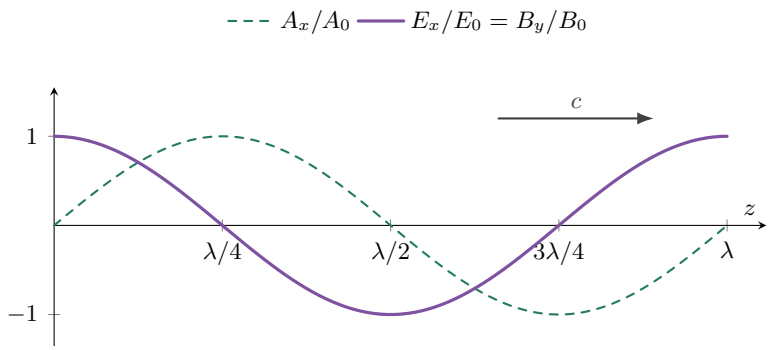
Solution. The vector-potential wave equation gives

$$(-k^2 + \omega^2/c^2)\mathbf{A} = 0 \quad \Longrightarrow \quad \omega = ck.$$

Let $\psi = kz - \omega t$. Differentiation gives

$$\boxed{\mathbf{E} = \omega A_0 \cos\psi\, \hat{\mathbf{x}}, \qquad \mathbf{B} = kA_0 \cos\psi\, \hat{\mathbf{y}}.} \tag{5.1}$$

Thus $E_0 = \omega A_0$, $B_0 = kA_0$, and $E_0 = cB_0$. The fields are in phase and perpendicular to each other and to $+z$. A point of fixed phase moves as $z = (\omega/k)t + \text{constant} = ct + \text{constant}$.



Spatial profiles at $t = 0$, with $\lambda = 2\pi/k$. The derivatives shift the potential profile by a quarter cycle; the physical electric and magnetic fields peak together.

Check / physical lesson

For $\mathbf{E} = E_x(z,t)\hat{\mathbf{x}}$ and $\mathbf{B} = B_y(z,t)\hat{\mathbf{y}}$, the curl equations reduce to

$$\partial_z E_x = -\partial_t B_y, \qquad -\partial_z B_y = c^{-2}\partial_t E_x.$$

Equation (5.1) satisfies both precisely when $\omega = ck$. This checks the result using the fields, independently of the potential equation.

From description to transport. We next ask how much energy and momentum these fields carry. Unlike the potentials, those quantities will be written directly in terms of the physical fields.

6 Field energy: begin with work on charges

We now derive a local energy balance from Maxwell's equations. For a charge q moving at velocity $\mathbf{v}$, the rate of work is

$$q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}.$$

The magnetic force is perpendicular to the velocity and does no work. Adding the electric work over all charge carriers gives

$$\boxed{\text{power delivered to matter per volume} = \mathbf{J} \cdot \mathbf{E}.} \tag{6.1}$$

This includes mechanical work and heating. A negative value means that matter supplies energy to the field.

Combine the two curl equations

Dot Ampère–Maxwell with $\mathbf{E}/\mu_0$ and Faraday with $\mathbf{B}/\mu_0$:

$$\frac{\mathbf{E} \cdot (\nabla \times \mathbf{B})}{\mu_0} = \mathbf{J} \cdot \mathbf{E} + \varepsilon_0 \mathbf{E} \cdot \partial_t \mathbf{E}, \qquad \frac{\mathbf{B} \cdot (\nabla \times \mathbf{E})}{\mu_0} = -\frac{\mathbf{B} \cdot \partial_t \mathbf{B}}{\mu_0}.$$

Use $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$. The difference gives

$$\nabla \cdot \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} = -\mathbf{J} \cdot \mathbf{E} - \partial_t \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right).$$

The factors $1/2$ arise because $\mathbf{E} \cdot \partial_t \mathbf{E} = \partial_t (E^2/2)$, and similarly for $\mathbf{B}$.

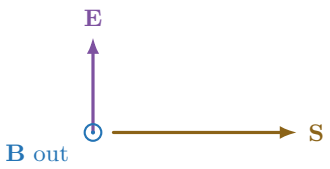
Define the energy per volume and energy flux:

$$\boxed{u = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}, \qquad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.} \tag{6.2}$$

Their units are $[u] = \mathrm{J/m^3}$ and $[\mathbf{S}] = \mathrm{W/m^2}$. The vector $\mathbf{S}$ is the **Poynting vector**. The balance becomes

$$\boxed{\partial_t u + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}.} \tag{6.3}$$

This is **Poynting's theorem**. It applies to static and changing fields, not only to waves.



direction: $\mathbf{E} \times \mathbf{B}$
magnitude: $EB \sin\theta/\mu_0$

The energy-flux direction is perpendicular to both fields. Here θ is their angle. Gold arrows show energy flow.

Physical meaning

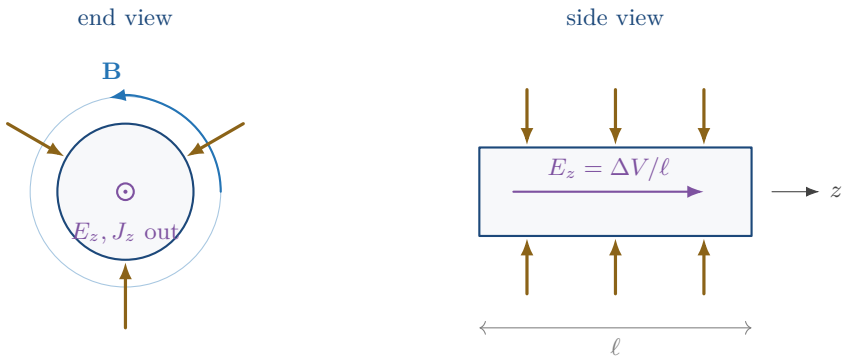
Field energy in a small region decreases when energy flows outward or when the field does work on matter. Energy can instead build up when more enters than leaves. The following examples distinguish steady conversion into heat from growing stored field energy.

7 Energy through a boundary: a resistor receives power

Integrate Eq. (6.3) over a fixed volume V. With outward normal $\hat{\mathbf{n}}$ and stored field energy $U_{\mathrm{EM}} = \int_V u\,\mathrm{d}V$,

$$\boxed{\frac{\mathrm{d}U_{\mathrm{EM}}}{\mathrm{d}t} + \oint_{\partial V} \mathbf{S}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\int_V \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V.} \tag{7.1}$$

Thus inward field power equals the increase of stored field energy plus the power delivered to matter. We now calculate this balance for a steady current.



A cylindrical resistor carries current along $+z$. Electric energy enters through its curved side and becomes heat inside. End effects are neglected.

Example — derive the input power from the surface fields

A uniform resistor has radius a, length ℓ, conductivity σ_c, and voltage drop $\Delta V > 0$. Find its resistance and the inward Poynting flux.

Solution. Ohm's local law gives

$$J_z = \sigma_c\frac{\Delta V}{\ell}, \qquad I = J_z\pi a^2, \qquad \mathcal{R} = \frac{\Delta V}{I} = \frac{\ell}{\sigma_c\pi a^2}.$$

At the surface, the tangential electric field is continuous and $B_\phi(a) = \mu_0 I/(2\pi a)$. Since $\hat{\mathbf{z}}\times\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\varrho}}$, the radial energy-flux component is

$$S_\varrho(a) = -\frac{E_z B_\phi(a)}{\mu_0} = -\frac{I\Delta V}{2\pi a\ell}.$$

The outward normal is $+\hat{\boldsymbol{\varrho}}$. Therefore

$$\boxed{P_{\mathrm{in}} = -S_\varrho(a)(2\pi a\ell) = I\Delta V = I^2\mathcal{R}.}$$

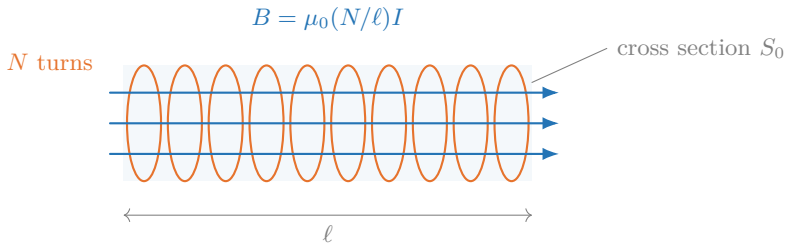
Independently, the volume heating rate is $\int_V \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V = J_z E_z \pi a^2\ell = I\Delta V$.

Check / physical lesson

The fields are steady, so $\mathrm{d}U_{\mathrm{EM}}/\mathrm{d}t = 0$. Nonzero energy flow does not require changing fields: here all incoming power is continuously transferred to matter.

8 Magnetic energy: derive the solenoid's inductance

We now turn stored field energy into a circuit quantity. Take a long solenoid of length ℓ, cross-sectional area S_0, and N turns. Neglect end effects and take the field to be uniform inside and negligible outside.



The magnetic energy occupies the solenoid's field-filled volume $S_0\ell$. The ideal model applies when the radius is small compared with the length.

Example — find inductance and stored energy from the same geometry

Let the current be I. Define the self-inductance $\mathcal{L}$ by the total flux linked by all turns: $N\Phi_B = \mathcal{L}I$. Find $\mathcal{L}$ and the magnetic energy.

Solution. Each turn links $\Phi_B = BS_0$. Hence

$$N\Phi_B = N\left(\mu_0 \frac{N}{\ell} I\right) S_0 \quad \Longrightarrow \quad \boxed{\mathcal{L} = \frac{\mu_0 N^2 S_0}{\ell}.}$$

Integrating the energy density over the field region gives

$$\boxed{U_B = \frac{B^2}{2\mu_0} S_0 \ell = \frac{1}{2}\frac{\mu_0 N^2 S_0}{\ell} I^2 = \frac{1}{2}\mathcal{L}I^2.} \tag{8.1}$$

At fixed geometry, doubling I doubles B and quadruples the stored energy. At fixed I, doubling N also quadruples the energy.

Independent check: build the current slowly

Faraday's law gives the self-induced emf $\mathcal{E}_{\text{self}} = -\mathcal{L}\dot{I}$. For an ideal lossless winding, an external source supplies the opposing emf $\mathcal{E}_{\text{drive}} = +\mathcal{L}\dot{I}$. Its input power and total work are

$$P_{\text{in}} = \mathcal{E}_{\text{drive}} I = \mathcal{L} I \dot{I}, \qquad U_B = \int_0^I \mathcal{L} I' \, \mathrm{d}I' = \frac{1}{2}\mathcal{L}I^2.$$

The source-work calculation agrees with the field-energy integral.

Physical meaning

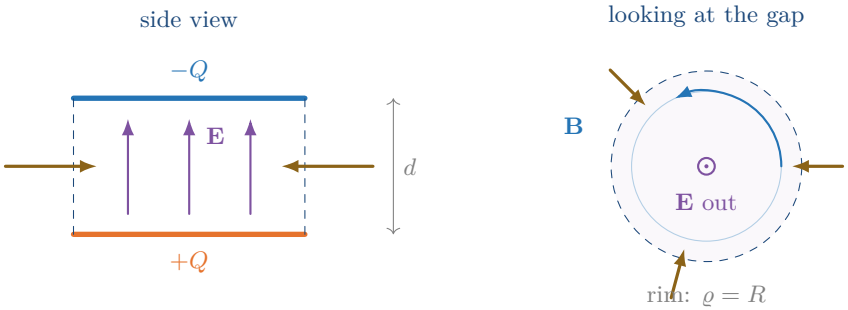
The electric counterpart is the capacitor from Week 4. For plate area S_p, gap d, and $E = \Delta V/d$,

$$U_E = \frac{\varepsilon_0 E^2}{2} S_p d = \frac{1}{2}\underbrace{\left(\frac{\varepsilon_0 S_p}{d}\right)}_{C} (\Delta V)^2.$$

Capacitance and inductance summarize how geometry stores electric and magnetic field energy. The next question is how that energy reaches the field region.

9 A charging capacitor: energy enters through the side

Reuse the circular capacitor of Week 7, with radius R, gap $d \ll R$, and charges $\pm Q(t)$. Let $I = \dot{Q}$ and choose $+z$ along the gap field. We calculate the power entering the gap. Use slow charging and neglect fringing, so the leading stored energy is electric.



For $Q > 0$ and $I > 0$, $\mathbf{E}$ is axial, $\mathbf{B}$ is azimuthal, and $\mathbf{S}$ points radially inward. The dashed line marks the integration boundary.

Example — recover charging power without following charge carriers

Find the inward field power in terms of Q, I, R, d.

Solution. Gauss's law and Ampère–Maxwell give, for $\varrho \le R$,

$$E = \frac{Q}{\varepsilon_0 \pi R^2}, \qquad B_\phi(\varrho) = \frac{\mu_0 I \varrho}{2\pi R^2}.$$

At the rim, $\hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\varrho}}$, so

$$\mathbf{S}(R) = -\frac{EI}{2\pi R}\hat{\boldsymbol{\varrho}}.$$

Multiply by the cylindrical side area $2\pi R d$ and reverse the outward sign:

$$\boxed{P_{\text{in}} = EId = I\Delta V = \frac{QI\,d}{\varepsilon_0 \pi R^2}.} \tag{9.1}$$

The stored electric energy independently gives

$$U_E = \frac{\varepsilon_0 E^2}{2}\pi R^2 d = \frac{Q^2 d}{2\varepsilon_0 \pi R^2}, \qquad \frac{\mathrm{d}U_E}{\mathrm{d}t} = \frac{QI\,d}{\varepsilon_0 \pi R^2} = P_{\text{in}}.$$

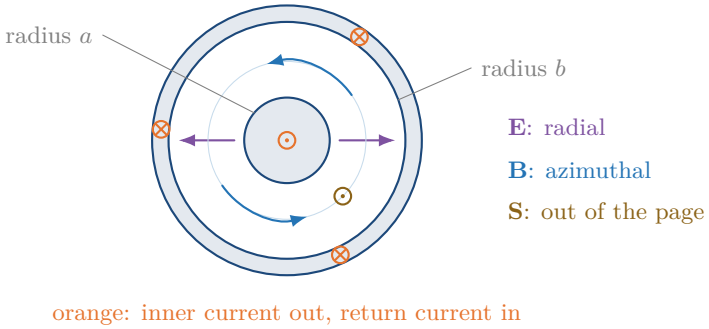
There is no conduction current and no $\mathbf{J} \cdot \mathbf{E}$ term in the vacuum gap.

Check / physical lesson

If $Q > 0$ but $I < 0$, the energy flux reverses: the capacitor releases its stored energy. For general time dependence, these formulas retain the leading slowly varying fields. The field-variation time must be long compared with R/c; magnetic-energy and propagation corrections are then small.

10 A coaxial cable: integrate the power carried by the fields

Take a long ideal coaxial cable with inner radius a, outer conductor's inner radius b, and vacuum in $a < \varrho < b$. The inner conductor is higher in potential by $\Delta V > 0$. Current $I > 0$ travels along $+z$ on the inner conductor and returns on the outer one. We seek the energy flux in the gap, away from the ends.



The electric and magnetic fields are both in the annular cross section. Their cross product carries energy along the cable through the vacuum gap.

Example — derive the fields, then integrate their cross product

Find the total power in terms of $a, b, I, \Delta V$.

Solution. Let λ be the inner charge per unit length. Gauss's law gives $E_\varrho = \lambda/(2\pi\varepsilon_0\varrho)$. Integrating from a to b gives $\Delta V = \lambda \ln(b/a)/(2\pi\varepsilon_0)$. Thus

$$\mathbf{E} = \frac{\Delta V}{\varrho \ln(b/a)}\hat{\boldsymbol{\varrho}}, \qquad \mathbf{B} = \frac{\mu_0 I}{2\pi\varrho}\hat{\boldsymbol{\phi}},$$

where the magnetic field follows from Ampère's law. Since $\hat{\boldsymbol{\varrho}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$,

$$\mathbf{S} = \frac{I\Delta V}{2\pi\varrho^2 \ln(b/a)}\hat{\mathbf{z}}.$$

A thin annulus has area $2\pi\varrho\,\mathrm{d}\varrho$, so

$$\boxed{P = \int_a^b S_z\, 2\pi\varrho\,\mathrm{d}\varrho = \frac{I\Delta V}{\ln(b/a)}\int_a^b \frac{\mathrm{d}\varrho}{\varrho} = I\Delta V.} \tag{10.1}$$

Physical meaning

The familiar circuit power is an integrated field flux. Changing a and b changes the local distribution of $\mathbf{S}$, but at fixed I and ΔV the total remains $I\Delta V$. Ideal conductors guide the fields; the gap carries the power.

11 Field momentum: energy flow has a direction

We now calculate the momentum stored in an electromagnetic field. In vacuum its density and total value in a volume are

$$\boxed{\mathbf{g} = \varepsilon_0 \mathbf{E} \times \mathbf{B} = \frac{\mathbf{S}}{c^2}, \qquad \mathbf{P}_{\mathrm{EM}} = \int_V \mathbf{g}\, \mathrm{d}V.} \tag{11.1}$$

The relation follows from Maxwell's equations and the Lorentz force law through the momentum balance in Sec. 13; its derivation is given in Sec. 18. The units of $\mathbf{g}$ are momentum per volume, $\mathrm{kg\,m^{-2}s^{-1}}$.

Example — energy and momentum of a plane wave

Take the wave of Sec. 5: $\mathbf{E} = E_0 \cos\psi\, \hat{\mathbf{x}}$, $\mathbf{B} = (E_0/c)\cos\psi\, \hat{\mathbf{y}}$. Calculate u, $\mathbf{S}$, and $\mathbf{g}$.

Solution. The electric and magnetic contributions are equal:

$$u_E = \frac{\varepsilon_0 E_0^2}{2}\cos^2\psi = u_B, \qquad u = \varepsilon_0 E_0^2 \cos^2\psi.$$

Their cross product gives

$$\boxed{\mathbf{S} = cu\,\hat{\mathbf{z}}, \qquad \mathbf{g} = \frac{u}{c}\hat{\mathbf{z}}.} \tag{11.2}$$

The average of $\cos^2\psi$ over one period is $1/2$, so the **intensity**, or average power per area, is

$$\boxed{\mathcal{I} \equiv \langle S_z \rangle = \frac{1}{2}\varepsilon_0 c E_0^2.}$$



Both fields reverse every half cycle. Their cross product keeps the same direction: energy and momentum continue along $+z$.

Example — momentum of a travelling pulse

A plane pulse carries energy U in a chosen transverse area, with edge effects neglected. Its local fields obey $B = E/c$ and travel along $+z$.

Solution. Integrating $\mathbf{g} = (u/c)\hat{\mathbf{z}}$ over that part of the pulse gives

$$\boxed{\mathbf{P}_{\mathrm{pulse}} = \frac{U}{c}\hat{\mathbf{z}}.}$$

Stopping or reflecting this momentum exerts a force. We will calculate that force after introducing field stress.

12 Maxwell stress: the surface direction matters

Energy flux has one scalar quantity to transport. Momentum has three components, and any component can cross a surface with any orientation. We therefore need a matrix, the **Maxwell stress tensor**, to keep track of both directions:

$$\boxed{T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right).} \tag{12.1}$$

Here $i, j \in \{x, y, z\}$, and δ_{ij} is 1 when $i = j$ and 0 otherwise. For a surface with unit normal $\hat{\mathbf{n}}$, the **traction** is

$$\mathbf{t} = \mathbf{T} \cdot \hat{\mathbf{n}}, \qquad t_i = T_{ix} n_x + T_{iy} n_y + T_{iz} n_z.$$

Traction has units of force per area. The first index labels its component; the second labels the surface-normal direction.

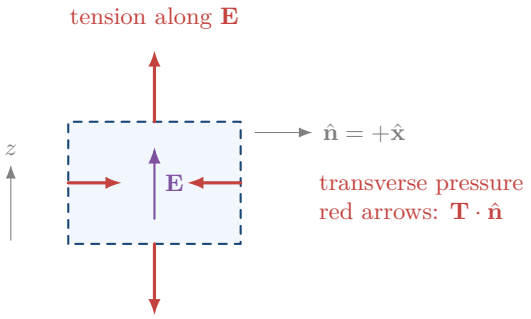
Example — read stress in a uniform electric field

Let $\mathbf{E} = E\hat{\mathbf{z}}$ and $\mathbf{B} = 0$. What traction acts across faces normal and parallel to the field?

Solution. With $u_E = \varepsilon_0 E^2/2$,

$$\mathbf{T} = u_E \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A face with $\hat{\mathbf{n}} = +\hat{\mathbf{z}}$ has $\mathbf{t} = +u_E\hat{\mathbf{z}}$. A face with $\hat{\mathbf{n}} = +\hat{\mathbf{x}}$ has $\mathbf{t} = -u_E\hat{\mathbf{x}}$. The stress has tension along the field and pressure in transverse directions.



A section through an imaginary volume in a uniform field. Opposite faces have opposite normals and opposite tractions. The total force on the closed volume is zero.

Physical meaning

A nonzero stress on a face is not, by itself, the net force on an object. We must add the tractions over a closed boundary and account for changing field momentum inside. If the field is tilted, an off-diagonal component such as $T_{xz} = \varepsilon_0 E_x E_z$ also gives tangential traction on a face normal to z.

13 Momentum balance: boundary transfer and field storage

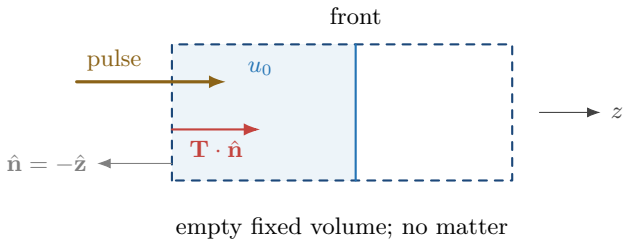
We now connect stress to the measurable Lorentz force density $\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$. Maxwell's equations give

$$\boxed{\mathbf{f} = \nabla \cdot \mathbf{T} - \partial_t \mathbf{g}, \qquad (\nabla \cdot \mathbf{T})_i = \sum_j \partial_j T_{ij}.} \tag{13.1}$$

The derivation is in Sec. 18. For a fixed volume, the divergence theorem gives

$$\boxed{\mathbf{F}_{\text{matter}} = \oint_{\partial V} \mathbf{T} \cdot \hat{\mathbf{n}}\, \mathrm{d}S - \frac{\mathrm{d}\mathbf{P}_{\text{EM}}}{\mathrm{d}t}.} \tag{13.2}$$

The force on matter equals momentum supplied through the boundary minus momentum accumulating in the field. With this stress convention, the *outward momentum-flux tensor is* $-\mathbf{T}$. This sign follows directly by writing $\partial_t \mathbf{g} + \nabla \cdot (-\mathbf{T}) = -\mathbf{f}$.



A pulse enters an empty volume. Boundary momentum transfer initially increases field momentum inside the volume; it does not imply a force on matter.

Example — why the field-momentum term cannot be omitted

An ideal plane pulse enters a box through area S_0, before its front reaches the exit. Its flat part has energy density u_0 and travels along $+z$. Find both terms in Eq. (13.2).

Solution. A transverse plane wave has $E_z = B_z = 0$, so $T_{zz} = -u_0$. At the entrance $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$, giving

$$\oint \mathbf{T} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = u_0 S_0 \hat{\mathbf{z}}.$$

In time $\mathrm{d}t$, the pulse fills additional volume $S_0 c\, \mathrm{d}t$. Since $\mathbf{g} = (u_0/c)\hat{\mathbf{z}}$,

$$\frac{\mathrm{d}\mathbf{P}_{\text{EM}}}{\mathrm{d}t} = \frac{u_0}{c} S_0 c \hat{\mathbf{z}} = u_0 S_0 \hat{\mathbf{z}}.$$

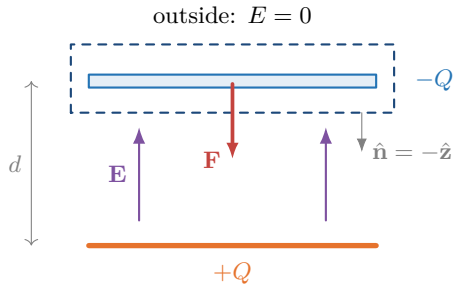
The terms cancel: $\mathbf{F}_{\text{matter}} = 0$, as they must in empty space.

Method / strategy

Choose a closed surface where the fields are simple. Evaluate $\mathbf{T} \cdot \hat{\mathbf{n}}$ using its *outward* normal. Drop the field-momentum derivative only when it vanishes. Static fields and a steady periodic average with no net momentum buildup are common cases.

14 Capacitor attraction: calculate the force from the gap field

Take parallel plates of area S_p, separation d, and charges $\pm Q$. Neglect edge fields. We calculate the force on the upper, negatively charged plate. In the gap, $\mathbf{E} = E\hat{\mathbf{z}}$ with $E = Q/(\varepsilon_0 S_p)$; the exterior field is zero.



The dashed pillbox encloses the upper plate. Its lower face lies in the gap and has outward normal $-\hat{\mathbf{z}}$; its upper face sees zero field.

Example — derive the pressure and its parameter dependence

Find the attractive pressure and force, then express the answer using either Q or ΔV.

Solution. The gap has $T_{zz} = \varepsilon_0 E^2/2$. The fields are static, so only the stress integral contributes:

$$\mathbf{F}_{\text{upper}} = -\frac{1}{2}\varepsilon_0 E^2 S_p \hat{\mathbf{z}}.$$

The attractive pressure and force magnitude are therefore

$$\boxed{p = \frac{\varepsilon_0 E^2}{2}, \qquad F = \frac{Q^2}{2\varepsilon_0 S_p} = \frac{\varepsilon_0 S_p}{2}\frac{(\Delta V)^2}{d^2}.} \tag{14.1}$$

At fixed Q and S_p, the ideal-plate force is independent of d. At fixed ΔV and S_p, it decreases as d^{-2}.

Two physical checks

Force from the other plate. One plate produces $E_{\text{other}} = Q/(2\varepsilon_0 S_p) = E/2$. A plate does not exert a net force on itself, so $F = QE_{\text{other}} = Q^2/(2\varepsilon_0 S_p)$. Using the full gap field in QE would count twice the correct force.

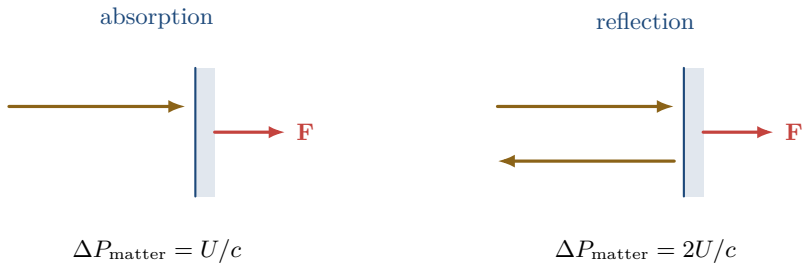
Energy at fixed charge. The stored energy is $U_E = Q^2 d/(2\varepsilon_0 S_p)$. The generalized force in the direction of increasing separation is

$$F_d = -\left(\frac{\partial U_E}{\partial d}\right)_Q = -\frac{Q^2}{2\varepsilon_0 S_p}.$$

The negative sign means attraction. At fixed voltage, the source's work must also be included in an energy-based force calculation.

15 Radiation pressure: reverse momentum to increase the force

We now use $P_{\text{pulse}} = U/c$ to calculate a mechanical force without tracking microscopic charges. Let a plane beam of intensity $\mathcal{I}$ strike a stationary flat surface normally. The illuminated area is S_0, so the incident power is $P = \mathcal{I}S_0$.



An absorber stops the incident field momentum. A mirror reverses it. The incident and reflected arrows are separated vertically for readability; incidence is normal.

Example — an opaque surface with arbitrary reflectivity

A fraction η of the incident energy is reflected directly backward and the remaining fraction $1-\eta$ is absorbed. Neglect thermal re-emission during the measurement. Find the average force and pressure.

Solution. In time Δt, the incident energy is $U = \mathcal{I}S_0\Delta t$. Its momentum is $+U\hat{\mathbf{z}}/c$; the reflected momentum is $-\eta U\hat{\mathbf{z}}/c$. The surface receives the difference:

$$\Delta\mathbf{P}_{\text{matter}} = (1+\eta)\frac{U}{c}\hat{\mathbf{z}}.$$

Divide by time and area:

$$\boxed{\mathbf{F} = (1+\eta)\frac{\mathcal{I}S_0}{c}\hat{\mathbf{z}}, \qquad p = (1+\eta)\frac{\mathcal{I}}{c}.} \tag{15.1}$$

Thus a perfect absorber has $p = \mathcal{I}/c$, and a perfect mirror has $p = 2\mathcal{I}/c$. Using $\mathcal{I} = \varepsilon_0 c E_0^2/2$, these become $\varepsilon_0 E_0^2/2$ and $\varepsilon_0 E_0^2$.

Check / physical lesson

For a perfect mirror, the tangential electric fields cancel at the surface while the magnetic fields add: $B_{\text{surface}} = 2B_{\text{incident}}$. The time-averaged stress pressure is

$$\left\langle \frac{B_{\text{surface}}^2}{2\mu_0} \right\rangle = \frac{B_0^2}{\mu_0} = \frac{2\mathcal{I}}{c},$$

where $B_0 = E_0/c$ is the incident magnetic-field amplitude. This independently confirms the reflected-momentum result.

Physical meaning

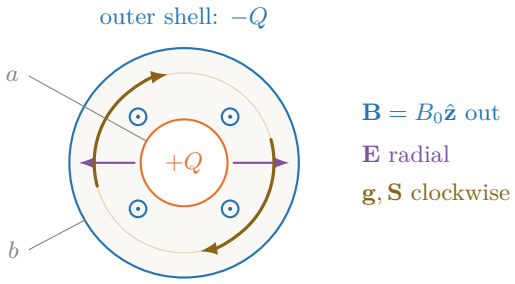
A stationary perfect mirror absorbs no net power but still feels a force. Force measures momentum transfer; absorbed power measures energy transfer. The mirror does no mechanical work while its velocity is zero.

16 Angular momentum: stored in a static field

About a fixed origin, the field angular-momentum density and total are

$$\boxed{\boldsymbol{\ell}_{\mathrm{EM}} = \mathbf{r} \times \mathbf{g}, \qquad \mathbf{L}_{\mathrm{EM}} = \int_V \mathbf{r} \times \mathbf{g}\, \mathrm{d}V.} \tag{16.1}$$

We now reuse the coaxial electric field, but replace the current-generated azimuthal magnetic field by an applied axial field. This changes the direction of the stored momentum.



Two long charged cylindrical shells, viewed with $+z$ out of the page. The electric and magnetic fields are static, yet their cross product circulates.

Example — field angular momentum between charged shells

Shells of radii $a < b$ and length $\ell \gg b$ carry uniform charges $+Q$ and $-Q$. An applied field $B_0\hat{\mathbf{z}}$ fills the gap. Neglect ends. Find $L_{\mathrm{EM},z}$.

Solution. Gauss's law gives $\mathbf{E} = Q\hat{\boldsymbol{\varrho}}/(2\pi\varepsilon_0\ell\varrho)$ for $a < \varrho < b$. Since $\hat{\boldsymbol{\varrho}} \times \hat{\mathbf{z}} = -\hat{\boldsymbol{\phi}}$,

$$g_\phi = -\frac{QB_0}{2\pi\ell\varrho}, \qquad (\mathbf{r} \times \mathbf{g})_z = \varrho g_\phi.$$

Integrate over annular shells of volume $2\pi\varrho\ell\,\mathrm{d}\varrho$:

$$\boxed{L_{\mathrm{EM},z} = \int_a^b \varrho g_\phi (2\pi\varrho\ell)\,\mathrm{d}\varrho = -\frac{QB_0}{2}(b^2 - a^2).} \tag{16.2}$$

The total linear momentum vanishes by azimuthal symmetry, but the angular-momentum contributions add.

Physical meaning

Static fields need not have zero momentum density or angular momentum. Here the energy flux forms closed circles, so it does not change the stored energy. Slowly switching off the magnetic field transfers this angular momentum to the charged shells; Sec. 20 calculates their rotation explicitly.

For an isolated field–matter system, time-translation symmetry corresponds to energy conservation, spatial-translation symmetry to momentum conservation, and rotational symmetry to angular-momentum conservation. The field contributions are essential parts of those totals.

Bridge to Week 9. We next develop wave propagation and polarization, then examine how the universal speed c connects space and time.

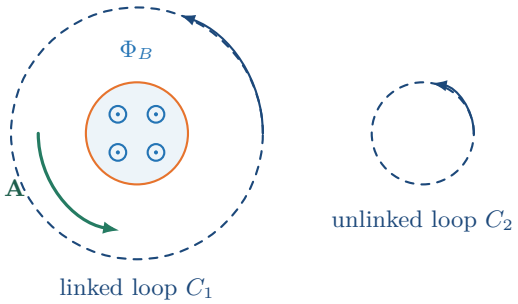
OPTIONAL — Gauge freedom and the global geometry of a loop
Further detail, following the main development.

17 A field-free region can retain a flux circulation

We now make precise the local–global distinction from Weeks 6 and 7. Outside an ideal long solenoid of radius R, the magnetic field vanishes, but an exterior vector potential is

$$\mathbf{A} = \frac{\Phi_B}{2\pi\varrho}\hat{\boldsymbol{\phi}}, \qquad \varrho > R,$$

where Φ_B is the solenoid's axial magnetic flux. Its local curl is zero because ϱA_ϕ is independent of ϱ.



A spanning surface for C_1 crosses the magnetic region. The smaller loop C_2 can span a surface entirely outside it.

Example — compare a linked and an unlinked loop

Find $\oint \mathbf{A}\cdot \mathrm{d}\boldsymbol{\ell}$ for the two positively oriented loops. Can a single-valued gauge function make $\mathbf{A} = 0$ everywhere outside the solenoid?

Solution. For the circular linked loop,

$$\oint_{C_1} \mathbf{A}\cdot \mathrm{d}\boldsymbol{\ell} = 2\pi\varrho\frac{\Phi_B}{2\pi\varrho} = \Phi_B.$$

For C_2, Stokes' theorem uses a surface where $\mathbf{B} = 0$, so its circulation is zero. Under any smooth, single-valued χ,

$$\oint_C \nabla\chi\cdot \mathrm{d}\boldsymbol{\ell} = \chi(\text{end}) - \chi(\text{start}) = 0.$$

The nonzero circulation around C_1 therefore cannot be removed by such a global gauge transformation.

Why a tempting local choice fails globally

The angular coordinate ϕ obeys $\nabla\phi = \hat{\boldsymbol{\phi}}/\varrho$ locally. One might choose

$$\chi = -\frac{\Phi_B}{2\pi}\phi \quad\Longrightarrow\quad \nabla\chi = -\mathbf{A}.$$

But after one turn, $\chi(\phi + 2\pi) - \chi(\phi) = -\Phi_B$. This is not a single-valued scalar on the whole exterior region. The choice works only on a patch with an angular cut, not globally around the hole.

Check / physical lesson

If the flux changes, a loop winding m times around the solenoid has $\mathcal{E} = -m\dot{\Phi}_B$. Local zero curl outside the solenoid does not remove the linked flux. For Stokes' theorem, the fields and potentials must be defined on the entire chosen surface.

OPTIONAL — The short derivation behind the momentum balance
Further detail, following the main development.

18 From Maxwell's equations to stress

We now verify the stress formula and its sign, rather than introducing a separate force law. The essential step is one vector identity. For any smooth vector field $\mathbf{F}$,

$$\nabla \cdot \left(\mathbf{F}\mathbf{F} - \frac{1}{2} F^2 \mathbf{1} \right) = \mathbf{F}(\nabla \cdot \mathbf{F}) + (\nabla \times \mathbf{F}) \times \mathbf{F}. \tag{18.1}$$

Here $(\mathbf{F}\mathbf{F})_{ij} = F_i F_j$, and $\mathbf{1}$ is the identity matrix. This notation is only a compact way to write the components already used in Eq. (12.1).

Check the identity by the product rule

For component i, summing over $j = x, y, z$,

$$\partial_j \left(F_i F_j - \frac{1}{2} \delta_{ij} F^2 \right) = F_i \partial_j F_j + F_j \partial_j F_i - F_j \partial_i F_j.$$

The first term is $F_i \nabla \cdot \mathbf{F}$. The last two are component i of $(\mathbf{F} \cdot \nabla)\mathbf{F} - \nabla(F^2/2) = (\nabla \times \mathbf{F}) \times \mathbf{F}$.

Apply it once to each field

For the electric contribution, Gauss's law and Faraday's law give

$$\nabla \cdot \mathbf{T}_E = \varepsilon_0 \mathbf{E}(\nabla \cdot \mathbf{E}) + \varepsilon_0 (\nabla \times \mathbf{E}) \times \mathbf{E} = \rho \mathbf{E} + \varepsilon_0 \mathbf{E} \times \partial_t \mathbf{B}.$$

For the magnetic contribution, use $\nabla \cdot \mathbf{B} = 0$ and Ampère–Maxwell:

$$\nabla \cdot \mathbf{T}_B = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{J} \times \mathbf{B} + \varepsilon_0 \partial_t \mathbf{E} \times \mathbf{B}.$$

Adding them combines the last two terms into a single time derivative:

$$\nabla \cdot \mathbf{T} = \underbrace{\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}}_{\mathbf{f}} + \partial_t \underbrace{(\varepsilon_0 \mathbf{E} \times \mathbf{B})}_{\mathbf{g}}.$$

Therefore $\mathbf{f} = \nabla \cdot \mathbf{T} - \partial_t \mathbf{g}$, with exactly the stress and momentum density used above.

18.1 The same calculation gives a torque balance

Multiply the force balance by $\mathbf{r}\times$ and integrate over a fixed volume. Because $T_{ij} = T_{ji}$, the interior couple terms cancel in pairs, such as $T_{xy} - T_{yx} = 0$. The remaining terms are

$$\boxed{\boldsymbol{\tau}_{\text{matter}} = \oint_{\partial V} \mathbf{r} \times (\mathbf{T} \cdot \hat{\mathbf{n}})\, \mathrm{d}S - \frac{\mathrm{d}\mathbf{L}_{\text{EM}}}{\mathrm{d}t}.} \tag{18.2}$$

Here $\boldsymbol{\tau}_{\text{matter}} = \int_V \mathbf{r} \times \mathbf{f}\, \mathrm{d}V$, about the same fixed origin. The symmetry of the stress tensor is what allows angular momentum to obey this balance without an extra local torque term.

Physical meaning

Energy balance uses a scalar density u and a vector flux $\mathbf{S}$. Momentum balance uses a vector density $\mathbf{g}$ and a tensor flux $-\mathbf{T}$. The added index records which momentum component crosses which surface; it does not introduce a different physical principle.

OPTIONAL — A useful bound on electromagnetic energy flow
Further detail, following the main development.

19 How large can the Poynting flux be?

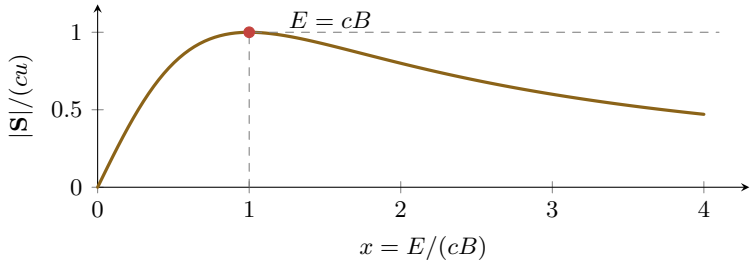
We now extract a general result from the energy density, without assuming a wave. For any vacuum fields, $|\mathbf{E}\times\mathbf{B}| \le EB$. Also,

$$cu - \frac{EB}{\mu_0} = \frac{c\varepsilon_0}{2}(E-cB)^2 \ge 0.$$

Combining these facts gives

$$\boxed{|\mathbf{S}| \le cu.} \tag{19.1}$$

For nonzero fields, equality requires $\mathbf{E}\perp\mathbf{B}$ and $E = cB$. A single plane wave meets both conditions. Parallel fields can store energy while giving zero Poynting flux.



For perpendicular fields with $B \neq 0$, the ratio is $2x/(1+x^2)$. Equal electric and magnetic energy densities maximize the flux for a given total density.

Example — apply the bound to the coaxial cable

Find the stored energy per unit length U' of the vacuum coax in Sec. 10. When does its power reach cU'?

Solution. Integrate the two energy densities over the annulus:

$$U'_E = \int_a^b \frac{\varepsilon_0 E^2}{2}\,2\pi\varrho\,\mathrm{d}\varrho = \frac{\pi\varepsilon_0(\Delta V)^2}{\ln(b/a)}, \qquad U'_B = \int_a^b \frac{B^2}{2\mu_0}\,2\pi\varrho\,\mathrm{d}\varrho = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a}.$$

Since $P = I\Delta V$, the bound gives $P \le c(U'_E + U'_B)$ for positive $I\Delta V$. Equality requires $E = cB$ throughout the gap, or

$$\boxed{\frac{\Delta V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\ln\frac{b}{a}.}$$

At this ratio, $U'_E = U'_B$ and $P = cU'$. It is the voltage-to-current ratio of a single travelling wave in this ideal coax. Arbitrary steady values of I and ΔV need not have this ratio.

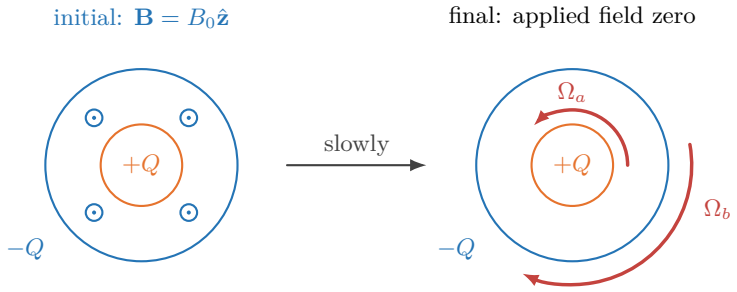
Physical meaning

The bound concerns energy flux relative to stored energy. It does not say that every static field pattern moves at $|\mathbf{S}|/u$. Fields may contain stored energy, counter-propagating waves, or circulating energy flow.

OPTIONAL — Make the stored angular momentum mechanically visible
Further detail, following the main development.

20 Switch off the field: the charged shells rotate

Use the two long shells of Sec. 16, now made of insulating material so their charges stay attached. They start at rest and can rotate freely about z. The applied field is uniform over both disks bounded by the shells; slowly reduce it from $B_0\hat{\mathbf{z}}$ to zero. Work to leading order in the shell charge: neglect the magnetic field generated by the shells' motion, as well as end effects and radiation.



With $Q > 0$ and $B_0 > 0$, the inner shell turns counterclockwise and the outer shell clockwise, viewed with $+z$ out of the page. The outer shell receives the larger angular impulse.

Example — derive the angular impulses without using field momentum

Find each shell's change of angular momentum directly from Faraday's law.

Solution. Both shell radii lie inside the region of uniform applied field, so

$$2\pi\varrho E_\phi = -\pi\varrho^2\dot{B} \quad\Longrightarrow\quad E_\phi = -\frac{\varrho}{2}\dot{B}.$$

The torque is charge times tangential field times radius:

$$\tau_{a,z} = -\frac{Qa^2}{2}\dot{B}, \qquad \tau_{b,z} = +\frac{Qb^2}{2}\dot{B}.$$

Integrating from B_0 to zero gives

$$\boxed{\Delta L_{a,z} = +\frac{Qa^2B_0}{2}, \qquad \Delta L_{b,z} = -\frac{Qb^2B_0}{2}.}$$

Adding the two results,

$$\boxed{\Delta L_{\text{matter},z} = -\frac{QB_0}{2}(b^2 - a^2) = L^{\text{initial}}_{\text{EM},z}.} \tag{20.1}$$

This agrees with the independent field integral in Eq. (16.2).

If the moments of inertia are $\mathcal{J}_a$ and $\mathcal{J}_b$, the final angular velocities, to the same approximation, are

$$\Omega_a = \frac{Qa^2B_0}{2\mathcal{J}_a}, \qquad \Omega_b = -\frac{Qb^2B_0}{2\mathcal{J}_b}.$$

Physical meaning

The electromagnetic field stored angular momentum before either shell moved. The induced electric field provides the torque that transfers it into mechanical rotation. The larger shell has a larger angular impulse because both the induced field and the lever arm grow with radius.