Electrodynamics & Relativity

Week 9: From Maxwell Waves to Relativity

One light signal, two observers

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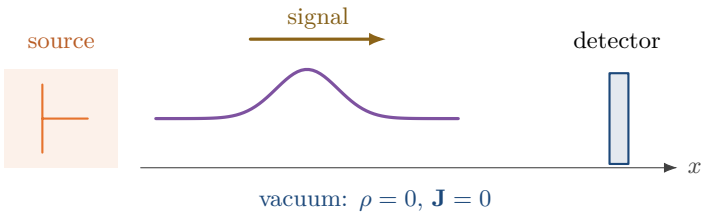
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Module 2025-CO-483 BSc Physics and Data Science

The question for this week. How does a light signal travel through empty space, and how can moving observers agree on its speed?
We will first build a travelling electromagnetic field, then calculate what it carries, and finally describe the same signal from a moving frame. The main route ends in Sec. 16. Four optional sections then give further derivations and applications.

1 From Maxwell's equations: a signal in empty space

Weeks 7 and 8 gave us the field equations and the energy carried by fields. We now apply them to the vacuum between a transmitter and a detector. Here ρ is charge density and $\mathbf{J}$ is current density. The transmitter's charges and currents prepare the disturbance. After it leaves that region, its evolution is governed by the source-free equations.



The curve represents a field profile along the line of travel. It is not the path of a particle.

Key relations

In this vacuum region, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{1.1}$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \qquad \partial_t \mathbf{E} = c^2 \nabla \times \mathbf{B}, \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}. \tag{1.2}$$

Here ∂_t means a time derivative at a fixed position. The divergence laws constrain the allowed field pattern. The curl laws tell us how that pattern changes with time: spatial circulation of one field determines the time derivative of the other.

Physical meaning

No sources here does not mean no fields here. A wave can enter from elsewhere, or already be present initially. Maxwell's equations evolve those fields; they do not require a charge at every point the signal passes.

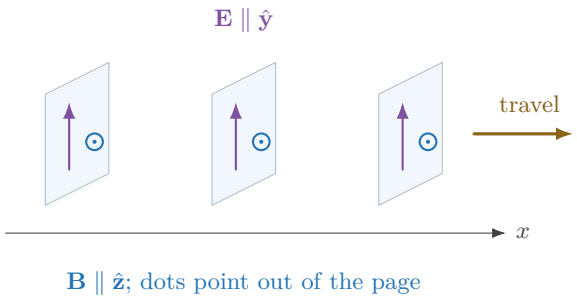
Next step. Choose the simplest geometry so that we can see this evolution directly.

2 Plane symmetry: find the wave equation

We first calculate the propagation law. Choose a disturbance travelling along x, with the same fields at all points of each transverse yz plane. For one transverse field direction, write

$$\mathbf{E} = E(x,t)\hat{\mathbf{y}}, \qquad \mathbf{B} = B(x,t)\hat{\mathbf{z}}.$$

The scalar functions E and B are signed components, not always positive magnitudes. An ideal **plane wave** has no variation across a transverse plane.



Plane symmetry removes the y and z derivatives. The field amplitudes may vary along x.

The divergence laws hold because E_y has no y dependence and B_z has no z dependence. The two curl laws reduce to

$$\boxed{\partial_t B = -\partial_x E, \qquad \partial_t E = -c^2 \partial_x B.} \tag{2.1}$$

For example, $(\nabla \times \mathbf{B})_y = -\partial_x B_z$, which fixes the second minus sign.

Eliminate one field, then the other

Differentiate the second relation with respect to time, and use the first:

$$\partial_t^2 E = -c^2 \partial_x (\partial_t B) = c^2 \partial_x^2 E.$$

Reversing the roles gives $\partial_t^2 B = c^2 \partial_x^2 B$. Thus

$$\boxed{\partial_x^2 E - \frac{1}{c^2}\partial_t^2 E = 0, \qquad \partial_x^2 B - \frac{1}{c^2}\partial_t^2 B = 0.} \tag{2.2}$$

These are **wave equations**: the spatial curvature of each profile determines its second time derivative.

Example — why a varying field along the travel direction fails

Could $\mathbf{E} = E_0 \cos(kx - ckt)\hat{\mathbf{x}}$, with $k > 0$, be a vacuum plane wave?

Solution. Each component obeys a wave equation, but $\nabla \cdot \mathbf{E} = -kE_0 \sin(kx - ckt) \neq 0$. It would require charge density $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$. The varying part of a vacuum plane wave must be **transverse**: perpendicular to its travel direction.

Uniform static background fields can be added separately; they are not the travelling disturbance.

Next step. Read the speed and shape of a solution to the wave equation.

3 A travelling profile: follow one feature

We now turn the wave equation into a moving picture. Let F be any sufficiently smooth profile, and set

$$\boxed{E(x,t) = F(x - ct).} \tag{3.1}$$

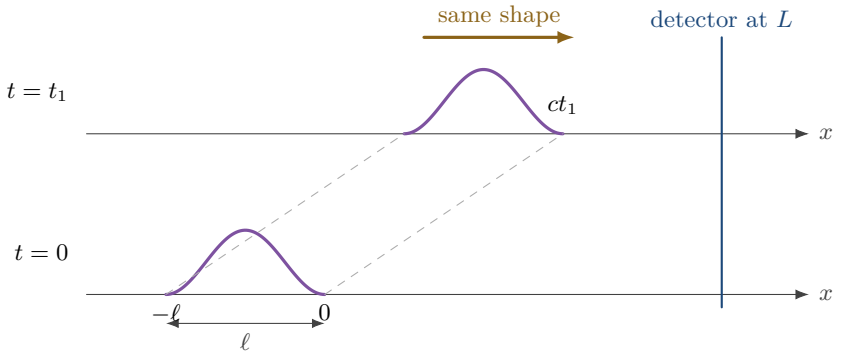
A point on the profile keeps the same value of its argument: $x - ct = \xi_0$, or $x = \xi_0 + ct$. Every feature therefore moves to the right at speed c.

A short substitution checks the whole profile

Writing $\xi = x - ct$, the chain rule gives

$$\partial_x E = F'(\xi), \quad \partial_x^2 E = F''(\xi), \qquad \partial_t E = -cF'(\xi), \quad \partial_t^2 E = c^2 F''(\xi).$$

Here primes mean derivatives of F with respect to its one argument. Equation (2.2) follows for every such profile, not only for a sine wave.



Two snapshots of one pulse. Its leading and trailing edges move by the same distance ct_1.

Example — when does a stationary detector receive the pulse?

At $t = 0$, a pulse occupies $-\ell \le x \le 0$. A detector is at $x = L > 0$. Find the arrival time and the time for the whole pulse to pass.

Solution. The leading edge is at $x = ct$ and the trailing edge at $x = ct - \ell$. Set each equal to L:

$$\boxed{t_{\text{first}} = \frac{L}{c}, \qquad t_{\text{last}} = \frac{L + \ell}{c}, \qquad \Delta t_{\text{pulse}} = \frac{\ell}{c}.} \tag{3.2}$$

Doubling L doubles the propagation delay but does not change the pulse duration.

Physical meaning

$F(x - ct)$ travels right; $G(x + ct)$ travels left. Their sum also solves the linear wave equation. For our main experiment we use one right-moving pulse. A finite leading edge makes the arrival of a new signal unambiguous.

Next step. A valid electromagnetic signal needs a magnetic field travelling with this electric profile.

4 Fields together: fix the magnetic field

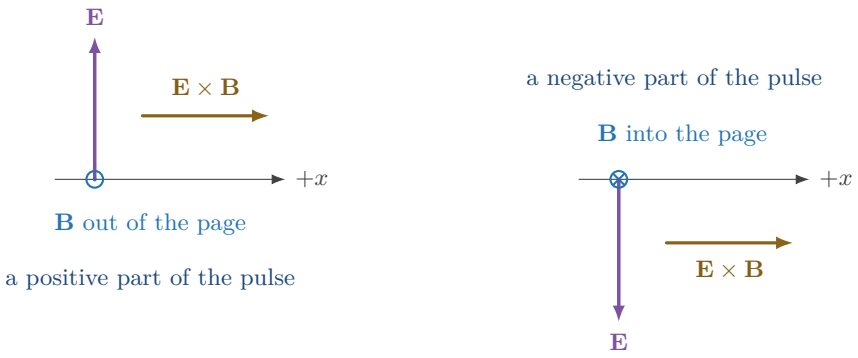
We now determine the magnetic field rather than choosing it independently. For $E = F(x - ct)$, try $B = bF(x - ct)$ with a constant coefficient b. The first reduced Maxwell equation gives

$$-cbF'(x - ct) = -F'(x - ct), \qquad b = \frac{1}{c}.$$

The second gives $-cF' = -c^2(F'/c)$, so it agrees. We have obtained a complete right-moving wave:

$$\boxed{\mathbf{E} = F(x - ct)\hat{\mathbf{y}}, \qquad \mathbf{B} = \frac{F(x - ct)}{c}\hat{\mathbf{z}}.} \tag{4.1}$$

The two fields have the same profile and reach their maxima at the same place and time. They are coupled parts of one wave.



The fields may both reverse. Their cross product still points along the travel direction, $+x$.

Example — reverse the direction of travel

Let $\mathbf{E} = G(x + ct)\hat{\mathbf{y}}$. Find $\mathbf{B}$ and check its direction.

Solution. Put $B = bG(x + ct)$ in $\partial_t B = -\partial_x E$: $cbG' = -G'$, so

$$\boxed{\mathbf{B} = -\frac{G(x + ct)}{c}\hat{\mathbf{z}}, \qquad \mathbf{E} \times \mathbf{B} = -\frac{G^2(x + ct)}{c}\hat{\mathbf{x}}.}$$

The magnetic sign changes when the travel direction changes.

Method / strategy

Check a proposed wave in two steps. First use the divergence laws to check the directions. Then use the curl laws to check the relative amplitude, sign, and time dependence. The wave equation alone is insufficient.

Physical meaning

The direction in which $\mathbf{E}$ oscillates is its **polarization direction**. For a fixed direction of oscillation, the wave is **linearly polarized**. Our choice is along y; rotating this direction is developed in optional Sec. 17.

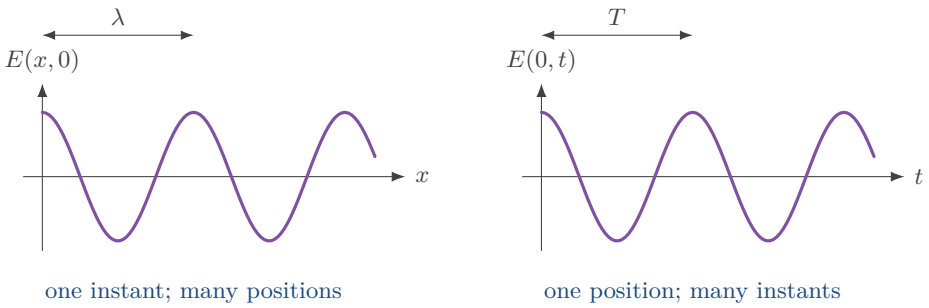
Next step. Describe a repeating profile using wavelength and frequency.

5 A harmonic wave: wavelength and frequency

We now use a sinusoidal profile to connect the wave's length and time scales. A **harmonic** wave is a sine or cosine wave with one frequency:

$$E(x,t) = E_0 \cos(kx - \omega t), \qquad k > 0. \tag{5.1}$$

The **phase** $kx - \omega t$ tells us where the field lies in its repeating cycle. Increasing phase by 2π leaves the field unchanged.



The same wave has a spatial period λ and a time period T. These are different measurements.

Quantity	Meaning	Relation
λ	distance between successive crests	$k = 2\pi/\lambda$
T	time between crests at one detector	$\omega = 2\pi/T$
f	cycles passing a detector per second	$f = 1/T$

Here k is the **wavenumber** (radians per metre), and ω is the **angular frequency** (radians per second). For the phase to be a function of $x - ct$, we need

$$\boxed{\omega = ck, \qquad \frac{\lambda}{T} = f\lambda = c.} \tag{5.2}$$

This relation also follows by substituting the cosine into the wave equation.

Example — two detectors see the same crest at different times

Two stationary detectors are separated by d along x. If the first records $E_1(t) = E_0 \cos(\omega t)$, what does the second record?

Solution. The profile takes d/c to reach the second detector, so

$$\boxed{E_2(t) = E_1(t - d/c) = E_0 \cos[\omega(t - d/c)]\,.}$$

The delay is d/c and the phase lag is $\omega d/c = kd$. For $d = \lambda/2$, the fields are opposite at the same instant; for $d = \lambda$, the records differ by one whole period.

Next step. Use the electric and magnetic fields to calculate the energy carried by the wave.

6 Energy: what a detector receives

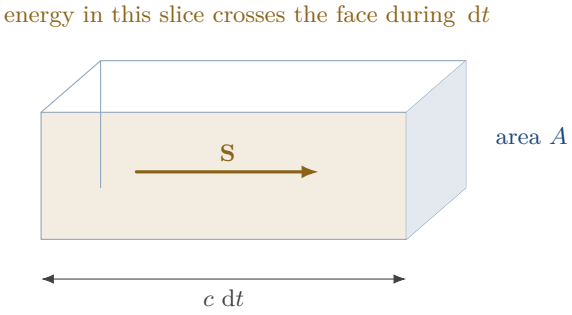
We now calculate what a detector can receive. Recall the energy per unit volume u and the energy flux $\mathbf{S}$ from Week 8:

$$u = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}, \qquad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}.$$

An energy flux is energy crossing unit area per unit time; $\mathbf{S}$ has units $\mathrm{W/m^2}$. For our single right-moving plane wave, $B = E/c$ and $\mu_0\varepsilon_0 c^2 = 1$ give

$$\boxed{u_E = u_B, \qquad u = \varepsilon_0 E^2, \qquad \mathbf{S} = cu\,\hat{\mathbf{x}}.} \tag{6.1}$$

Equal electric and magnetic energy does not mean energy alternates between the fields: the two contributions rise and fall together as the profile passes.



A slice of volume $Ac\,\mathrm{d}t$ contains energy $uAc\,\mathrm{d}t$. Dividing by $A\,\mathrm{d}t$ gives $S = cu$.

For a sinusoidal wave, $E = E_0\cos(kx - \omega t)$. The average of $\cos^2$ over a whole cycle is $1/2$. Define the **intensity** I as the cycle-averaged flux:

$$\boxed{I = \langle S_x \rangle = \frac{1}{2}\varepsilon_0 c E_0^2.} \tag{6.2}$$

Example — infer the field from measured power

A broad sinusoidal beam deposits average power P on a stationary absorbing detector of area A, perpendicular to the beam. The incident intensity is uniform over A. Find E_0, B_0, and the average energy density.

Solution. Since $P = IA$,

$$\boxed{E_0 = \sqrt{\frac{2P}{\varepsilon_0 cA}}, \qquad B_0 = \frac{E_0}{c}, \qquad \langle u \rangle = \frac{P}{Ac}.}$$

At fixed area, four times the power means twice the field amplitude.

Check / physical lesson

E_0 is the *peak* electric field. The factor $1/2$ belongs to the average of its square, not to the instantaneous relation $S = \varepsilon_0 c E^2$.

Next step. Use the pulse energy to find the momentum delivered to an absorber.

7 Momentum: a light pulse delivers an impulse

We now use the pulse energy to calculate its mechanical effect on a detector. For a right-moving plane pulse, uniform over area A, the energy received is

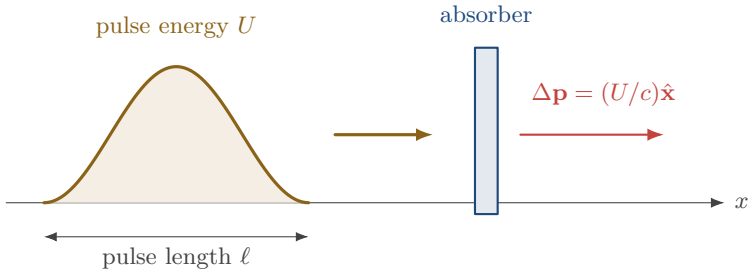
$$\boxed{U = A\int S_x\,\mathrm{d}t = \varepsilon_0 A\int F^2(\xi)\,\mathrm{d}\xi.} \tag{7.1}$$

The time integral is measured at the detector; the spatial integral spans the whole profile, with $\xi = x - ct$. They agree because the pulse moves at c. Optional Sec. 18 evaluates the integral for a specific pulse shape.

Week 8 gave the electromagnetic momentum density $\mathbf{g} = \mathbf{S}/c^2$. Using $\mathbf{S} = cu\hat{\mathbf{x}}$, we obtain $\mathbf{g} = (u/c)\hat{\mathbf{x}}$. Integration over the pulse then gives

$$\boxed{\mathbf{p}_{\text{pulse}} = \frac{U}{c}\hat{\mathbf{x}}.} \tag{7.2}$$

An absorbing detector, or its support, receives this momentum as an **impulse**: a change in mechanical momentum. We assume negligible recoil and no reflected pulse.



Energy arrives over a finite time. The total delivered momentum depends on the total energy, not on how quickly the pulse passes.

Example — same pulse energy, different duration

Two pulses each carry energy U to the same stationary absorber. One lasts for time T_p and the other for $2T_\mathrm{p}$. Compare their impulses and average forces.

Solution. Both deliver the same impulse, $\Delta p = U/c$. Average force is impulse divided by the delivery time, so

$$\boxed{\langle F_x\rangle_1 = \frac{U}{cT_\mathrm{p}}, \qquad \langle F_x\rangle_2 = \frac{U}{2cT_\mathrm{p}}.}$$

The longer pulse pushes half as strongly on average, but for twice as long. The total momentum transfer is unchanged. Here T_p denotes pulse duration, not the oscillation period T introduced in Sec. 5.

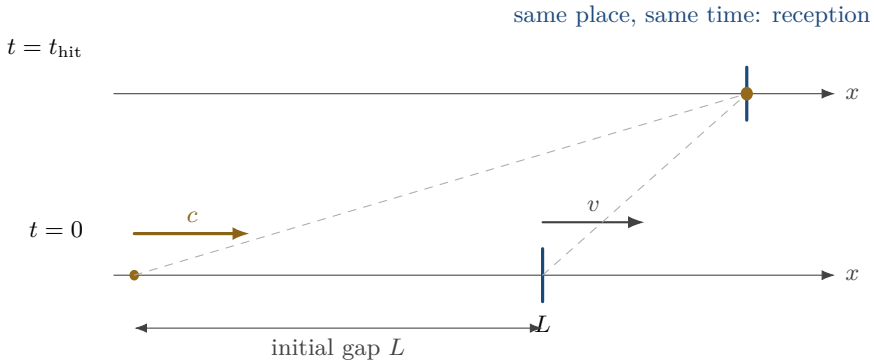
Physical meaning

The signal delivers energy and momentum through vacuum. No material medium needs to move from source to detector. For our single travelling plane wave, the energy and momentum travel in the same direction, with $p = U/c$.

Next step. Keep the same light signal, but now let its receiver move.

8 A moving detector: why time matters

We now keep the light signal along $+x$, and let the detector move in that direction with constant speed $0 < v < c$. At $t = 0$ the signal front is at $x = 0$ and the detector at $x = L$. All coordinates on this page are measured in the laboratory frame S.



The light front and detector meet. This calculation uses one set of laboratory positions and clocks.

Example — when and where does the light arrive?

Write the two positions and solve for their intersection.

Solution. The signal follows $x_{\text{light}} = ct$ and the detector follows $x_{\text{det}} = L + vt$. At reception,

$$ct_{\text{hit}} = L + vt_{\text{hit}} \quad \Longrightarrow \quad \boxed{t_{\text{hit}} = \frac{L}{c - v}, \qquad x_{\text{hit}} = \frac{cL}{c - v}.}$$

The gap decreases at the laboratory rate $c - v$. As $v \to c$, reception takes arbitrarily long.

Physical meaning

A closing rate is not yet a measurement in the detector's frame. The subtraction $c - v$ compares two positions at the same laboratory time. To find the light speed in a moving frame, we must compare distances and times measured using that frame's rulers and synchronized clocks.

Galilean kinematics would set $x' = x - vt$ and $t' = t$, giving light speed $c - v$. But if Maxwell's equations have the same form in every inertial frame, they predict the same vacuum speed c there.

The physical input for special relativity. The laws of physics have the same form in every inertial frame, and light in vacuum has the same speed c in those frames. These are physical principles, consistent with experiment; a wave equation in one frame alone does not prove them.

Next step. Define how a frame assigns a time and position to an event before changing coordinates.

9 Events and clocks: define the measurement

We now make the measurement procedure explicit. An **inertial frame** is a set of spatial coordinates and synchronized clocks at rest relative to one another, moving without acceleration. A single observer may use this clock network to assign coordinates.

An **event** is something occurring at one position and time: for example, a light front leaving $x = 0$, or reaching a detector. Its coordinates are (t, x, y, z). Different frames give different coordinates to the *same* event.

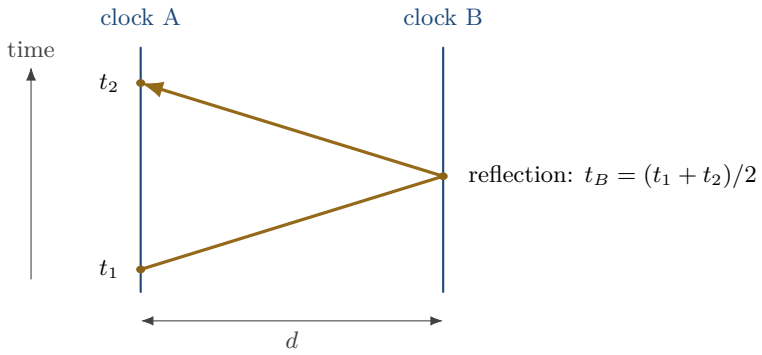
Example — synchronize a distant clock with light

Clock A sends a signal at its reading t_1. Clock B reflects it immediately, and A receives it at t_2. Both clocks are at rest in the frame. What time is assigned to reflection at B?

Solution. Using equal outward and return light travel times, assign

$$\boxed{t_B = \frac{t_1 + t_2}{2}.}$$

The separation is $d = c(t_2 - t_1)/2$. If the outward signal leaves at $t_1 = 0$, the reflection event has coordinates $(t_B, x_B) = (d/c, d)$.



A frame uses local clock readings at events. This diagram shows the order of events; its two axes are not drawn to the same scale.

Return to the moving detector. Call emission O and reception R. In the laboratory their coordinates are

$$O : (t, x) = (0, 0), \qquad R : (t, x) = \left(\frac{L}{c - v}, \frac{cL}{c - v}\right). \tag{9.1}$$

These are the two events we must transform. We do not replace R by the later event at which somebody sees a photograph of reception.

Method / strategy

Choose the moving frame S' to travel at speed $+v$ relative to S. Let the two coordinate origins coincide at event O, with $t = t' = 0$ there. The detector is stationary somewhere in S'; it need not be at the moving origin. Primes from this point denote *another frame*, not the gauge change used in Week 8.

Next step. Find the coordinate transformation that allows both clock networks to measure light speed c.

10 Lorentz transformations: compare event coordinates

We now need the coordinate rule that lets both inertial frames measure light speed c. The moving origin follows $x = vt$, so x' must depend on $x - vt$. Time must change as well: using the laboratory time in both frames would give $c - v$.

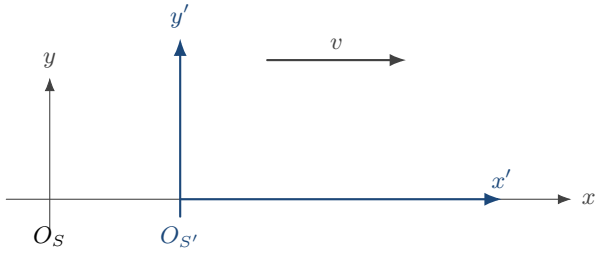
Key relations

For S' moving along $+x$ at speed v relative to S, with coincident origins at $t = t' = 0$,

$$\boxed{\begin{aligned} &x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \\ &y' = y, \qquad\qquad z' = z, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \end{aligned}} \tag{10.1}$$

The inverse is obtained by replacing v by $-v$ and interchanging primes.

These are the **Lorentz transformations**. We will use and check them here. Optional Sec. 20 derives their coefficients from linearity, the two light directions, and the equivalence of the frames.



parallel spatial axes
origins separate after $t = 0$

A change to a uniformly moving frame is called a Lorentz boost. The transverse coordinates y and z are unchanged.

Example — check both light directions

For $x = ct$, Eq. (10.1) gives

$$x' = \gamma(c - v)t, \qquad ct' = \gamma(c - v)t, \qquad \boxed{x' = ct'.}$$

For $x = -ct$, it gives $x' = -\gamma(c + v)t$ and $ct' = \gamma(c + v)t$, so $x' = -ct'$. Both directions retain speed c.

Physical meaning

The important new term is $-vx/c^2$: the time assigned to an event depends on where it happens. The same factor γ rescales the two mixed coordinates. For $v = 0$, $\gamma = 1$ and the frames agree exactly. For $|v| \ll c$, γ is close to one; when the time correction is also negligible, the Galilean relations are recovered as an approximation.

Method / strategy

Transform events, then interpret the result. Write down (t, x) for each event in S, apply both formulas, and only then calculate a speed or time difference in S'. Keeping t unchanged while transforming x mixes two frames in the same calculation.

Next step. Apply this rule to emission and reception in our moving-detector experiment.

11 The same signal: light speed in the moving frame

We now resolve the puzzle without changing the physical experiment. Transform the emission and reception events of Eq. (9.1) using Eq. (10.1). Remember that S' travels with the detector.

Example — one reception event, two coordinate descriptions

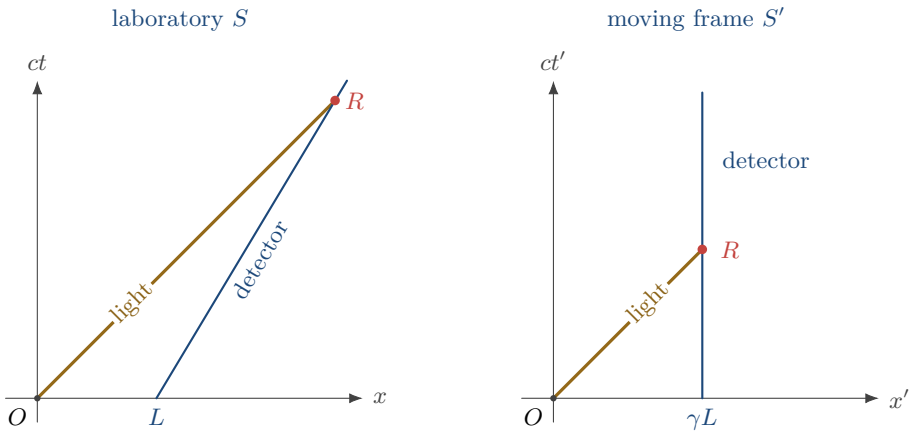
Emission is $O : (t, x) = (0, 0)$, so $t'_O = x'_O = 0$. At reception,

$$x'_R = \gamma\left(\frac{cL}{c-v} - v\frac{L}{c-v}\right) = \gamma L,$$
$$t'_R = \gamma\left(\frac{L}{c-v} - \frac{v}{c^2}\frac{cL}{c-v}\right) = \frac{\gamma L}{c}.$$

Consequently,

$$\boxed{\frac{x'_R - x'_O}{t'_R - t'_O} = \frac{\gamma L}{\gamma L/c} = c.} \tag{11.1}$$

The laboratory obtains c from x_R/t_R; the moving frame obtains c from x'_R/t'_R. Both coordinate descriptions refer to the same emission and reception.



A spacetime diagram places time vertically and position horizontally. A particle's history is its *worldline*. A stationary detector has a vertical worldline. Both panels use the same scale for distance and $c\times$ time, so light lies at 45°.

Physical meaning

The factor γL is the detector's fixed position in the particular S' coordinates we chose. Indeed, its whole laboratory worldline $x = L + vt$ gives $x' = \gamma L$. There is no rule that the detector must be at $x' = 0$.

What changed? The moving frame uses a different time coordinate as well as a different position coordinate. Keeping laboratory time while changing only the distance would reproduce the incorrect Galilean result $c - v$.

Next step. See what the position-dependent term in t' means for simultaneous events.

12 Simultaneity: compare separated events

We now isolate the consequence of the term $-vx/c^2$ in the Lorentz time transformation. For any two events, subtraction of their transformed coordinates gives

$$\boxed{\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right).} \tag{12.1}$$

Here Δ means "event B minus event A." Two events are **simultaneous** in a frame when that frame assigns them the same time.

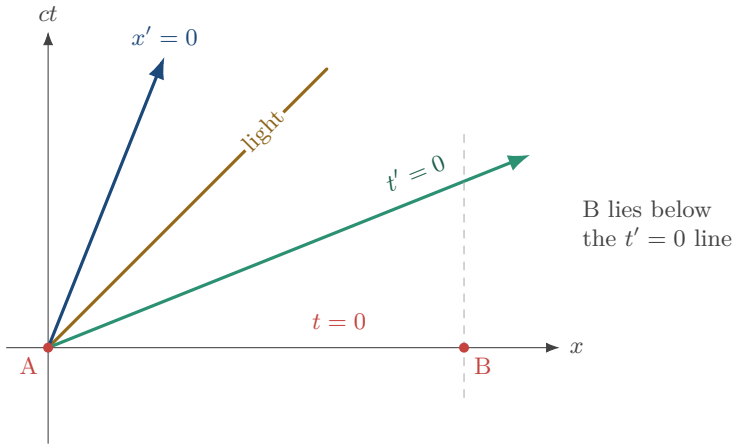
Example — two flashes simultaneous in the laboratory

Two lamps at $x = 0$ and $x = d > 0$ flash at laboratory time $t = 0$. Find the order of the flashes in S', which moves with $v > 0$.

Solution. Transform the emission events themselves:

	t	x	t'	x'
flash A	0	0	0	0
flash B	0	d	$-\gamma vd/c^2$	γd

Thus $\Delta t' = -\gamma vd/c^2 < 0$: the forward lamp B flashes earlier in S'. These are emission times assigned by synchronized clocks, not the times at which one observer sees the flashes.



The line of simultaneous events tilts when the frame changes. Flash B lies on $t = 0$, but has $t' < 0$.

The axes of the moving coordinates have simple definitions:

$$\underbrace{x' = 0 \iff x = vt}_{\text{moving origin's history}}, \qquad \underbrace{t' = 0 \iff ct = (v/c)x}_{\text{events simultaneous with A in } S'}.$$

The first is the ct' axis and the second the x' axis. They tilt toward the light direction as v increases; this is not an ordinary rotation of a paper diagram.

Check / physical lesson

If the events happen at the same position, $\Delta x = 0$, there is no spatial contribution to $\Delta t'$. If $v = 0$, the two frames agree on simultaneity. For separated events and nonzero relative motion, they need not agree.

Next step. Find a combination of time and distance that does remain the same.

13 The interval: what both observers agree on

We now calculate what two frames agree on even when their distances and times differ. For two events, define the **spacetime interval**

$$\boxed{\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2.} \tag{13.1}$$

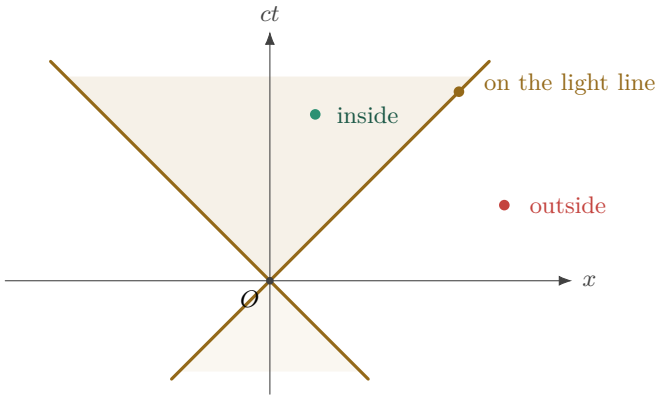
The symbol Δs^2 is a signed quantity; it can be positive, zero, or negative.

The mixed terms cancel

The Lorentz transformation gives

$$\begin{aligned} c^2\Delta t'^2 - \Delta x'^2 &= \gamma^2\left[c^2\left(\Delta t - \frac{v\Delta x}{c^2}\right)^2 - (\Delta x - v\Delta t)^2\right] \\ &= \gamma^2(1 - v^2/c^2)(c^2\Delta t^2 - \Delta x^2) \\ &= c^2\Delta t^2 - \Delta x^2. \end{aligned}$$

Since $y' = y$ and $z' = z$, we obtain $\boxed{\Delta s'^2 = \Delta s^2}$.



upper shaded cone: causal future
lower shaded cone: causal past

In one spatial dimension the light cone appears as two crossing lines. In three dimensions the future light front is a sphere of radius ct.

Example — can a signal connect two events?

Take $O : (0, 0)$ and $Q : (T, L)$ with $T > 0$ and $L > 0$. The required speed is L/T. Their interval is $c^2T^2 - L^2$:

	Interval	Name	Connection
$L < cT$	> 0	timelike	a slower-than-light signal is possible
$L = cT$	$= 0$	lightlike	a light signal can connect the events
$L > cT$	< 0	spacelike	no signal with speed $\leq c$ can connect them

These names describe the event separation. Every inertial frame agrees on its type because the interval is invariant.

Next step. Use a timelike interval to find the time recorded by one moving clock.

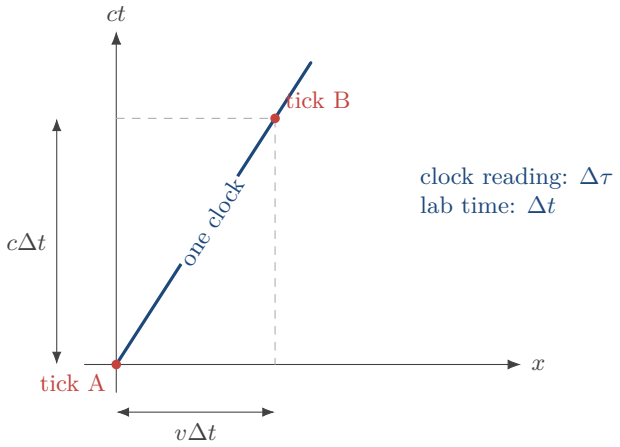
14 Proper time: one clock makes the journey

We now distinguish the time between two events in a coordinate system from the elapsed time recorded by one clock present at both events. The latter is the clock's **proper time**, denoted $\Delta\tau$.

Suppose a clock moves uniformly at speed v along x. Between two ticks, $\Delta x = v\Delta t$ in the laboratory. In the clock's rest frame both ticks occur at the same position, so $\Delta s^2 = c^2\Delta\tau^2$. Using the invariant interval gives

$$c^2\Delta\tau^2 = c^2\Delta t^2 - v^2\Delta t^2 \quad\Longrightarrow\quad \boxed{\Delta\tau = \Delta t\sqrt{1 - v^2/c^2} = \frac{\Delta t}{\gamma}.} \tag{14.1}$$

This is **time dilation**: the laboratory time between ticks of the moving clock is γ times the time read on that clock.



The two events lie on the history of one clock. Its elapsed time comes from the interval, not the Euclidean length of the sloping line.

Example — a clock carried between two laboratory stations

Stations A and B are at rest in the laboratory, separated by distance D. A clock travels from A to B at constant speed v. Find the elapsed laboratory time and the elapsed time on the travelling clock.

Solution. The laboratory gives $\Delta t = D/v$. The moving clock records

$$\boxed{\Delta\tau = \frac{D}{\gamma v} = \frac{D}{v}\sqrt{1 - \frac{v^2}{c^2}}.}$$

An independent check uses Eq. (12.1) with $\Delta x = D$:

$$\Delta t' = \gamma\left(\frac{D}{v} - \frac{vD}{c^2}\right) = \frac{D}{\gamma v} = \Delta\tau.$$

Both departure and arrival occur at the travelling clock, so $\Delta t'$ is its proper time.

Physical meaning

The laboratory uses synchronized clocks at two different stations. The traveller uses one clock. Stating which events and which clocks are involved removes the apparent ambiguity in "the elapsed time."

Next step. Apply the same coordinate transformation to a general particle velocity.

15 Velocities: transform distance and time together

We now obtain the velocity rule that includes ordinary particles as well as light. For two nearby events on a particle's path, divide the Lorentz coordinate differences:

$$u' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x - v\,\mathrm{d}t}{\mathrm{d}t - v\,\mathrm{d}x/c^2} \quad \Longrightarrow \quad \boxed{u' = \frac{u - v}{1 - uv/c^2}, \qquad u = \frac{\mathrm{d}x}{\mathrm{d}t}.} \tag{15.1}$$

The common factor γ cancels. The denominator accounts for the transformed time.

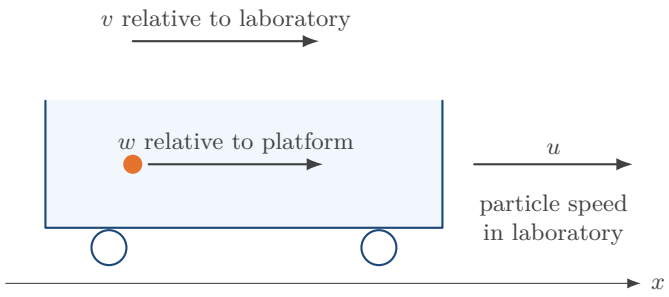
Example — launch a particle from a moving platform

A platform moves at speed $v > 0$ in the laboratory. It launches a particle forward at speed $w > 0$ measured in the platform frame, with $v, w < c$. Find the particle's laboratory speed u.

Solution. Set $u' = w$ in Eq. (15.1) and solve:

$$w(1 - uv/c^2) = u - v \quad \Longrightarrow \quad \boxed{u = \frac{v + w}{1 + vw/c^2}.}$$

For $v, w \ll c$, this approaches $v + w$. For $w = c$, it gives $u = c$ exactly.



Each speed is defined relative to a named frame. The formula combines the two relative motions.

Why two subluminal speeds cannot add to more than c

Subtract the result from c and factor:

$$c - u = \frac{c(1 + vw/c^2) - v - w}{1 + vw/c^2} = \frac{(c - v)(c - w)}{c(1 + vw/c^2)} > 0.$$

Both factors in the numerator are positive when $v, w < c$. Thus the laboratory speed also satisfies $u < c$.

Check / physical lesson

For a light signal with $u = c$, Eq. (15.1) gives $u' = c$. For a particle at rest in the laboratory, $u = 0$ gives $u' = -v$. The rule handles both limits using the same transformation.

Next step. Return to the fields: what must a moving observer call electric and magnetic?

16 Back to electromagnetism: why the fields must transform

We have followed one chain: Maxwell's equations permit a transverse signal travelling at c; the fields transport energy and momentum; preserving that speed between inertial frames requires space and time to transform together. We finish by asking what this implies for the fields themselves.

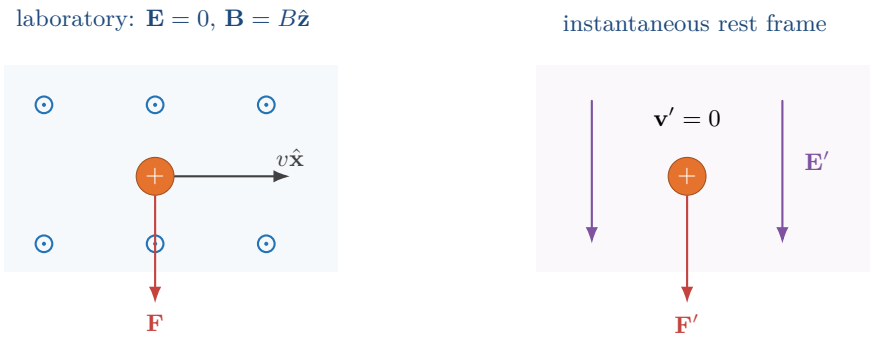
Example — a magnetic force becomes an electric-force question

In the laboratory, a positive charge q has instantaneous velocity $v\hat{\mathbf{x}}$ in a uniform magnetic field $B\hat{\mathbf{z}}$, with no electric field. What force acts on it? What must a frame moving with the charge describe at that instant?

Solution. In the laboratory,

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = -qvB\hat{\mathbf{y}}.$$

In the instantaneous rest frame of the charge, $\mathbf{v}' = 0$, so the magnetic part $q\mathbf{v}' \times \mathbf{B}'$ vanishes. The charge still accelerates; the force law there must contain an electric field: $\mathbf{F}' = q\mathbf{E}' \neq 0$. The transformation of the fields must therefore mix their electric and magnetic parts.



The same charge at the same event needs different field descriptions. The primed magnetic field is omitted from the right panel because its force on the momentarily stationary charge is zero.

We have inferred the need for $\mathbf{E}'$ without assuming $\mathbf{F}' = \mathbf{F}$. The force components also transform. The moving frame is comoving with the charge only at the chosen instant, since the magnetic field bends the charge's path.

What Week 10 will calculate. We now know how positions and times change between frames, and how one clock records proper time. Next we will use this structure to transform charge and current, potentials, and the electric and magnetic fields. The aim is to describe the same electromagnetic experiment consistently for different inertial observers.

Three checks before moving on. For a plane wave travelling along $+x$ with $\mathbf{E}$ along $+y$, the magnetic field points along $+z$. Its measured intensity determines the field amplitude through a *square*. To compare two observers, first identify the same events, then transform both their positions and their times. These are the working tools needed for the next step.

OPTIONAL — Rotating the wave's field directions
Further detail, following the main development.

17 Polarization: one transverse direction fixes the other

We now give the wave any fixed electric direction in the transverse plane. Let $\hat{\mathbf{n}}$ be the unit vector in the travel direction. Rotate the coordinates so that $\hat{\mathbf{n}}$ takes the role of $+\hat{\mathbf{x}}$. Our result then reads

$$\boxed{\hat{\mathbf{n}} \cdot \mathbf{E} = 0, \qquad \mathbf{B} = \frac{1}{c}\hat{\mathbf{n}} \times \mathbf{E}.} \tag{17.1}$$

Thus $\mathbf{E}$, $\mathbf{B}$, and $\hat{\mathbf{n}}$ are mutually perpendicular, and $|\mathbf{E}| = c|\mathbf{B}|$. This extends the fixed polarization used in the main text.

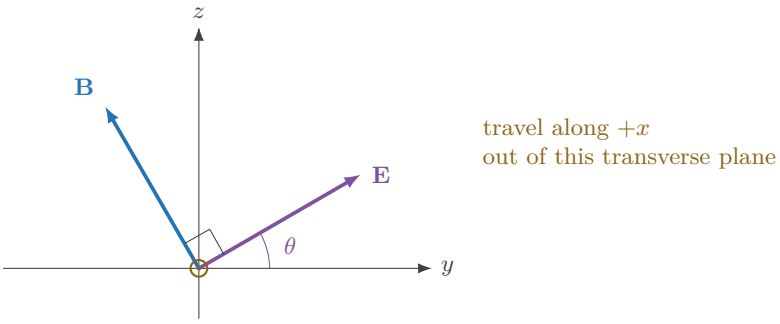
Example — rotate the electric direction, keeping the same travel direction

A wave travels along $+x$, has wavelength λ, and its electric direction makes angle θ with $+y$ towards $+z$. Construct both fields.

Solution. Let $k = 2\pi/\lambda$ and $\psi = k(x - ct)$. Then

$$\begin{aligned} \mathbf{E} &= E_0(\cos\theta\,\hat{\mathbf{y}} + \sin\theta\,\hat{\mathbf{z}})\cos\psi, \\ \mathbf{B} &= \frac{E_0}{c}(-\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}})\cos\psi. \end{aligned} \tag{17.2}$$

We used $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ and $\hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}$. For $\theta = 0$ this reduces to our original E_y, B_z pair.



view of the yz plane; field arrows show directions

Rotating $\mathbf{E}$ rotates $\mathbf{B}$ with it. The fields retain a right-handed arrangement with the travel direction.

Check the geometry without relying on the drawing

The transverse unit vectors in Eq. (17.2) obey

$$(\cos\theta\,\hat{\mathbf{y}} + \sin\theta\,\hat{\mathbf{z}}) \cdot (-\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) = 0,$$

and their cross product is $(\cos^2\theta + \sin^2\theta)\hat{\mathbf{x}} = \hat{\mathbf{x}}$. Each rotated field is a sum of two transverse Maxwell solutions, so linearity guarantees that the full pair also obeys the curl laws.

Physical meaning

A wavefront is a surface of equal phase. For a plane wave, the phase is $\mathbf{k} \cdot \mathbf{r} - \omega t$, where $\mathbf{k} = k\hat{\mathbf{n}}$ is the **wave vector**. The wavefronts are perpendicular to $\hat{\mathbf{n}}$; successive crest planes are separated by λ. Our choice $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ makes this simply $kx - \omega t$.

OPTIONAL — A complete energy integral for one pulse
Further detail, following the main development.

18 A finite profile: calculate energy and momentum explicitly

We now evaluate the general energy formula from Sec. 7. This calculation shows how the pulse shape enters U; it is not needed to use the main result $p = U/c$.

Example — a pulse with a definite beginning and end

Take $\mathbf{E} = F(x - ct)\hat{\mathbf{y}}$ and $\mathbf{B} = F(x - ct)\hat{\mathbf{z}}/c$, where

$$F(\xi) = \begin{cases} E_0 \sin^2(\pi\xi/\ell), & -\ell \le \xi \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

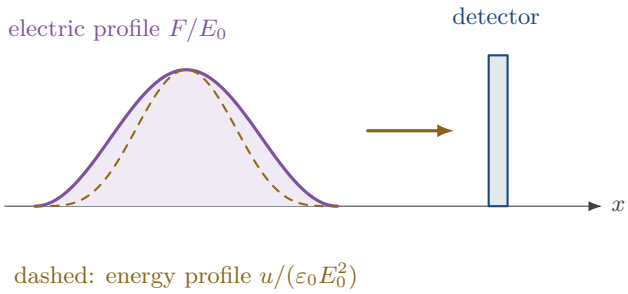
The profile and its first derivative join continuously to zero at each edge. Find the energy and momentum passing through a stationary detector of area A.

Solution. Insert the profile into Eq. (7.1):

$$U = \varepsilon_0 A E_0^2 \int_{-\ell}^{0} \sin^4(\pi\xi/\ell)\, \mathrm{d}\xi.$$

Use $\sin^4 q = (3 - 4\cos 2q + \cos 4q)/8$. The cosine terms integrate to zero over the interval, leaving

$$\boxed{U = \frac{3}{8}\varepsilon_0 A E_0^2 \ell, \qquad \mathbf{p}_{\text{pulse}} = \frac{3\varepsilon_0 A E_0^2 \ell}{8c}\hat{\mathbf{x}}.} \tag{18.1}$$



The energy profile is proportional to the square of the electric profile. It is more concentrated near the pulse centre.

An independent check: integrate the detector's time record

At $x = L$, the pulse arrives between $t = L/c$ and $t = (L + \ell)/c$. The measured energy flux is

$$S_x(L, t) = \varepsilon_0 c E_0^2 \sin^4\left[\frac{\pi(L - ct)}{\ell}\right]$$

within that interval, and zero outside. Set $q = \pi(L - ct)/\ell$, so $\mathrm{d}t = -\ell\, \mathrm{d}q/(\pi c)$:

$$A \int S_x(L, t)\, \mathrm{d}t = \frac{\varepsilon_0 A E_0^2 \ell}{\pi} \int_{-\pi}^{0} \sin^4 q\, \mathrm{d}q = \frac{3}{8}\varepsilon_0 A E_0^2 \ell.$$

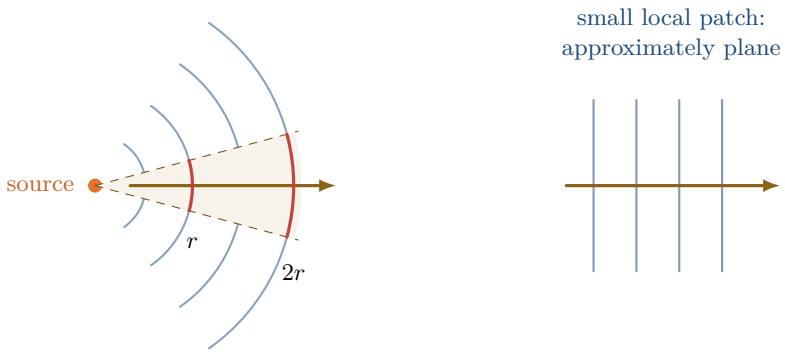
This agrees with the energy stored in the moving spatial profile.

OPTIONAL — From a plane wave to a distant source
Further detail, following the main development.

19 Spreading: separate dilution from propagation delay

We now explain how a detector farther from a localized source receives a later, usually weaker, signal. These are two different effects.

Far enough from the source, where the outward radiation dominates, a small part of a curved wavefront looks approximately planar. Its local fields obey the plane-wave relations derived above. Across larger distances, the wave spreads out.



A fixed angular bundle covers a larger area as it expands. The drawing shows a cross-section; the wave spreads in three dimensions.

The same energy crosses a larger area

For a steady radiation pattern, an angular bundle has area proportional to r^2. Conservation of outward power in that bundle gives

$$I(r)r^2 = \text{constant along a fixed direction}, \qquad I(r) \propto r^{-2}, \qquad E_0(r) \propto r^{-1}.$$

The amplitude relation follows from $I = \varepsilon_0 c E_0^2/2$ for a locally sinusoidal wave. Different directions can have different intensities; spherical wavefronts do not require isotropic emission.

Example — compare two detectors on the same outward ray

Place identical small detectors at r and ar, where $a > 1$ and both are in the radiation region. A source emits a sinusoidal wave. Compare the arrival of the same crest and the intensities at the two detectors.

Solution. The additional travel distance is $(a-1)r$, so

$$\boxed{\Delta t_{\text{arrival}} = \frac{(a-1)r}{c}, \qquad \frac{I(ar)}{I(r)} = \frac{1}{a^2}, \qquad \frac{E_0(ar)}{E_0(r)} = \frac{1}{a}.}$$

At twice the distance, the delay increases by r/c, the intensity is quartered, and the field amplitude is halved.

A field measured at time t and radius r depends on the source at the earlier **retarded time** $t - r/c$. This is the finite propagation delay already encountered with retarded potentials in Week 8. A sinusoid extending forever is a useful calculation tool; a pulse makes the arrival of information visible.

OPTIONAL — Deriving the coordinate transformation
Further detail, following the main development.

20 Lorentz coefficients: preserve both light directions

We now derive the transformation used in Sec. 10. Uniform space and time, together with uniform relative motion, lead to a linear coordinate rule. Since the moving origin follows $x = vt$, write

$$x' = a(x - vt), \qquad t' = bt + dx.$$

The coefficients depend on v; a, b are dimensionless and d has units of inverse speed.

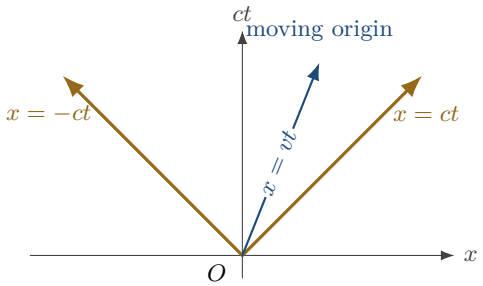
First fix how time mixes with position

Right-moving light obeys $x = ct$ and must obey $x' = ct'$. Left-moving light obeys $x = -ct$ and must obey $x' = -ct'$. Substituting gives

$$a(c - v) = c(b + dc), \qquad a(c + v) = c(b - dc).$$

Adding the equations gives $b = a$. Subtracting gives $d = -av/c^2$. Hence

$$x' = a(x - vt), \qquad t' = a(t - vx/c^2).$$



The transformation must preserve both boundaries of the light cone while assigning $x' = 0$ to the moving origin.

Then fix the common scale

The two frames are physically equivalent. Reversing the relative velocity therefore gives the inverse transformation. Spatial symmetry makes the common scale the same for v and $-v$, so $x = a(x' + vt')$. Substituting the expressions above,

$$x = a^2 \left[x - vt + v\left(t - \frac{vx}{c^2}\right)\right] = a^2(1 - v^2/c^2)x.$$

Thus $a^2(1 - v^2/c^2) = 1$. Choosing the positive solution connected continuously to $v = 0$ gives $a = \gamma = 1/\sqrt{1 - v^2/c^2}$. This fixes the longitudinal part of Eq. (10.1).

Example — check that the inverse also restores the time

Verify the inverse time rule $t = \gamma(t' + vx'/c^2)$.

Solution.

$$\gamma\left(t' + \frac{vx'}{c^2}\right) = \gamma^2\left[t - \frac{vx}{c^2} + \frac{v}{c^2}(x - vt)\right] = \gamma^2(1 - v^2/c^2)t = t.$$